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REFLECTIVE INTELLIGENCE AND MATHEMATICS LEARNING

by

© Donald Bruce Harrison

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Reflective Intelligence and Mathematics Learning," submitted by Donald Bruce Harrison in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

The purpose of the study was to gather theoretical and empirical evidence to provide a basis for appraising R. R. Skemp's theory of mathematics learning. Attention was particularly focused on assessing the importance of the operation of reflective intelligence, as defined by Skemp, in influencing success in school mathematics.

The theoretical part of the investigation took the form of a review of the literature that pointed out inadequacies of classical learning theories in explaining mathematics learning, described Piaget's theory of cognitive growth, showed that the theories of Bruner, Dienes, and Skemp have been built on a Piagetian foundation, and marshalled support for Skemp's theory of mathematics learning.

An empirical study was designed to investigate whether Skemp's measures of reflective intelligence have sufficient mathematics performance predictive value to significantly improve the predictive efficiency of the best available aptitudes battery. The influence of anxiety in affecting mathematics performance was also taken into account. In May and June, 1966, six classes of grade eight students were administered a battery of ten tests that included an aptitude test battery, a test anxiety

scale, Skemp's tests of reflective intelligence and a measure of mathematics performance. Stepwise multiple regression, partial correlation, and canonical correlation procedures were carried out on the data collected to test the major hypotheses of the study. In another phase of the investigation, two classes of students at each of the grade levels from five to eleven were administered Skemp's tests, and age level and sex contrasts were made in terms of mean levels of reflective intelligence. Two-way analyses of variance were carried out on the data collected.

It was found that adding scores from Skemp's tests to scores from the aptitude tests battery significantly improved the prediction of mathematics performance scores. That success in mathematics depends on the exercise of reflective intelligence, appears to have been confirmed. The influence of anxiety factors was not found to produce significant effects on mathematics performance in the present study. Fifteen and sixteen-year-olds were found to operate at higher levels of reflective intelligence than ten to twelve-year-olds.

In general, both from a theoretical and from an empirical point of view, the usefulness of Skemp's reflective intelligence construct in relation to mathematics learning was substantiated. A number of potentially useful implications for mathematics education were drawn from the findings of the study.

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D. B. H.

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THE NATURE AND SIGNIFICANCE OF THE PROBLEM

I. INTRODUCTION

In the past decade, a great deal of effort has been expended in the production of radically revised mathematics curricula, particularly in the United States. However, as one group of writers has observed:

> Efforts to remake the curriculum are all to the good, but curriculum revision strikes at only one side of the trouble in mathematical education. Much more needs to be known about the mental functioning of children who are expected to pass through the curriculum and about teachers who are expected to operate it.
>
> . . . .
>
> Too little is known about how the minds of children can be brought to cope with mathematical concepts. Too little is known about what it takes to make them want to use mathematics.[1]

It has been noted that remarkably little in the way of psychological research has been carried out to provide a better understanding of the mental processes involved in

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[1]Henry S. Dyer, Robert Kalin, and Frederick M. Lord, Problems in Mathematical Education (Princeton, New Jersey: Educational Testing Service, 1956), pp. 26-27.

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mathematics learning.[2] R. R. Skemp, a psychologist at the University of Manchester and a former mathematics scholar, believes that an important part of the reason for the sparseness of research into mathematics learning has been the lack of a theory of learning which is applicable to mathematics. Behaviourist learning theories, which have dominated psychological thinking for many years, have neglected the study of mental processes on the assumption that, since mental processes as such are not observable, they are not valid subject for scientific study. However,

> No physicist would refuse to study magnetic fields on the grounds that those were unobservable, and confine himself to the study of arrangements of iron filings. The criterion for the scientific respectability of an <u>observation</u> is that others can make it too; but for a scientific <u>concept</u>, the criteria are its powers to unify, to explain and to predict these observables. In psychology, concepts relating to mental processes are as capable as any others of satisfying the above criteria.[3]

Since in the learning of mathematics there is a hierarchical dependency of later learning stages on previous ones, any mathematics learning theory ". . . must deal not only with thinking processes but with a highly organized and interconnected system of thinking processes."[4]

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[2]R. R. Skemp, "The Psychology of Learning and Teaching Mathematics," Study No. 1 on the various aspects of the teaching of mathematics in secondary schools, Paris, UNESCO, 1962, p. 4. (Mimeographed, Limited Distribution)

[3]<u>Ibid</u>.

[4]<u>Ibid</u>., p. 5.

Generally, learning theories have not met this criterion since they have dealt ". . . with the learning of isolated facts and responses, rather than integrated systems of knowledge and skills."[5]

Skemp, strongly influenced by the works of Piaget and Inhelder in the areas of number, arithmetic, geometry and logic, has endeavoured to initiate efforts to remedy the situation described in the preceding paragraphs by formulating a three-part theory of mathematics learning. The main features of the theory are:

1. That mathematical learning depends on the reflective use of intelligence,
2. That there is an efficient method for forming mathematical concepts, and
3. That schematic learning is essential in the learning of mathematics.[6]

In order to stimulate research that would confirm or refute his conjectures, Skemp has generated more than thirty hypotheses from his basic theory.[7] His hope is that these hypotheses might be instrumental in ensuring that ". . . some of the future individual researches in this

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[5]Ibid., p. 4.

[6]R. R. Skemp, "A Three-Part Theory for Learning Mathematics," New Approaches to Mathematics Teaching, F. W. Land, editor (London: Macmillan and Co., Ltd., 1963), pp. 41-47.

[7]Skemp, "The Psychology of Learning and Teaching Mathematics," pp. 22-27.

field may be led to take forms such that the results can be integrated into a comprehensive theory of mathematical learning."[8]

One is surprised (or would be surprised if the phenomenon were not so common) at the apparent "inability" of some intelligent children to do well in mathematics. What does one need in addition to general intelligence to succeed in mathematics? What abilities must be developed before one can appreciate and use successively more abstract mathematical ideas? How can these abilities best be nurtured? How do school children think when confronted with mathematical situations? How can mathematics be taught to stimulate the development of independent, resourceful, and mathematically sound thinking in students? Can Skemp's theory provide a framework for the search for answers to the preceding questions?

In the course of a previous study,[9] a test of mathematical understandings was developed to measure how well grade eight students use the basic mathematical concepts and procedures they have learned. Correlations among scores

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[8] _Ibid_., p. 21.

[9] D. B. Harrison, "An Analysis of the Effectiveness of Three Mathematics Programs at the Grade Eight Level" (unpublished Master's Thesis, The University of Alberta, Edmonton, 1964).

obtained by such students on tests of arithmetic achievement, verbal ability, quantitative ability, and on the test of mathematical understandings indicated that success in using mathematical ideas in novel situations requires more than arithmetic skill, verbal ability, quantitative ability, and mere knowledge of the appropriate concepts. Could some of the variability in mathematics understanding scores be accounted for by abilities other than those measured in the experiment? Skemp's description of the processes involved in learning mathematics suggests answers to this question and to those in the preceding paragraph.

Skemp's conception of reflective intelligence, which is described on pages 11 to 14 of the present report, focuses attention on a way of using one's intelligence that has not traditionally been taken into account in attempts to describe mathematical ability (e.g., factor analytic studies[10]). Studying Skemp's work, one soon realizes that the reflective intelligence concept plays a fundamental role in his theory of mathematics learning. Since so many of the hypotheses that have been generated by Skemp stem from the notion of reflective intelligence, confirmation of its existence and

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[10]Jack Wrigley, "The Factorial Nature of Ability in Elementary Mathematics," British Journal of Educational Psychology, 28:61-78, 1958.

its relation to mathematics learning is of crucial importance to the theory. Consequently, the present study has been designed to investigate the relationship between reflective intelligence and mathematics performance. If the relevance of reflective intelligence in mathematics learning can be confirmed, it promises to be a useful construct for guiding procedures in mathematics teaching. It was with this basic idea in mind that the problem described in the following section was formulated.

II. THE PROBLEM AND ITS SIGNIFICANCE

The problem investigated can be characterized best in terms of the three questions that guided the design of the study. These questions are:

1. Is the presence of reflective intelligence a factor of enough importance for success in mathematics that the addition of measures of reflective intelligence to measures of general intelligence significantly improves the prediction of one's performance in mathematics?

2. What effects on the relationships between measured reflective intelligence and mathematics performance result from taking into account student anxiety toward testing situations?

3. Are there any significant differences among the mean levels of reflective intelligence exhibited by boys or girls in age categories from ten to sixteen?

The first question is fundamental to the development of Skemp's theory, and the need for such a theory of mathematics learning cannot be overstated. Furthermore, as will be seen in the discussion of the third question and in the discussion of reflective intelligence, demonstration of the importance of the presence of reflective intelligence in relation to mathematics learning would have significant implications for mathematics teaching procedures. While Skemp has shown that there is a positive correlation between reflective intelligence, as measured by his tests, and mathematics achievement,[11] the writer has felt that there is need for more conclusive evidence in terms of demonstrable mathematical understandings and in consideration of the contributions of general intelligence. A major purpose of the study being reported was to provide statistical tests to assess the relative importance of reflective intelligence as a factor in the demonstration of an understanding of mathematics.

Skemp's work on reflective intelligence suggests that when a child performs poorly in mathematics it may be either that he has not formed the necessary concepts and operations or that he cannot reflect on them, or both.

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[11]R. R. Skemp, "Reflective Intelligence and Mathematics," British Journal of Educational Psychology, 31:51-54, February, 1960.

It also suggests that it is reflective ability which is particularly liable to blockage by certain kinds of emotional disturbance (e.g., extreme anxiety). Perhaps this is why some very intelligent people have difficulty learning mathematics.[12] The second question posed on page 6 anticipates the interference of anxiety with reflective thinking in testing situations.

Sarason and his colleagues have found, through clinical experience, that test anxiety is not only frequent but that test situations frequently evoke strongly anxious responses. In our test-giving, test-conscious culture, students are well aware of the effects on their progress in life that accrue from their test performances. Considering that attitudes towards tests are relevant to performance in the test situation, and, hence, influence the variables under study, the effects of such attitudes should be taken into account.[13]

For the purpose of the study, and using Sarason's frame of reference, the anxious child is defined as one who is unduly concerned: about how he will do in school,

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[12] Ibid., p. 54.

[13] S. B. Sarason and others, Anxiety in Elementary School Children (New York: John Wiley and Sons, Incorporated, 1960), pp. 8-9.

about being wrong in test-like situations, about being promoted, and about not understanding what the teacher expects of him.[14]

The third question posed on page 6 is closely related to considerations of what kinds of teaching procedures might be appropriate in teaching mathematics to children at various age levels. At what stage would it seem feasible to make the transition from teaching that <u>lays the foundation</u> for formal mathematics to teaching that assumes student readiness to cope with formal mathematics? Inhelder and Piaget believe that formal thinking processes are not fully developed before adolescence[15] and that before this period children are not fully capable of formal logical thinking. On the other hand, Dienes[16] claims to have taught advanced mathematics topics to pre-adolescent children. In Skemp's words:

> Can such claims be justified? For it is important to know whether, following Piaget's view that pre-adolescents are not ready for formal logical processes, mathematics should be postponed till the secondary age group; or whether Dienes is right in considering that children can be successfully

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[14] *Ibid.*, pp. 10-11.

[15] Bärbel Inhelder and Jean Piaget, *The Growth of Logical Thinking from Childhood to Adolescence* (New York: Basic Books, Inc., 1958). English Edition.

[16] Z.P. Dienes, *An Experimental Study of Mathematics-Learning* (London: Hutchinson, 1963).

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> introduced much earlier to a much wider range of mathematical concepts than is at present customary.[17]

That the conflict of points of view here is not as great as it appears on the surface is discussed on pages 97 and 98 of the present report. The point is still well taken, though, in relation to plans for accommodating one's teaching approach to take into account the child's characteristic modes of thinking.

To further emphasize the importance of knowing how and when reflective intelligence arises, consider the fact that, especially in mathematics, a teacher is able to think in ways not fully available to his pupils. He may give explanations that are very clear to himself, or other adults, but that are unintelligible to children. The teacher's way of thinking, because it is more direct and precise than the child's, may appear most useful for clear and concise explanations. Consequently, the less a child understands his work, the more this unintelligible mode of explanation may be used. Intelligent self-correction requires the ability to become aware of relationships between purely mental events. Mathematically oriented teachers may so habitually make mental

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[17]Skemp, "The Psychology of Learning and Teaching Mathematics," p. 15.

self-corrections that they may fail to realize that it is difficult or even impossible for some of their pupils. "Learning to do something is one thing; but it appears that knowing <u>how</u> one does it is something new, which involves learning the same things all over again at a different level . . ."[18] If explanations are given which assume well developed reflective ability in the learner before it is there, such explanations will serve no useful purpose.

Before proceeding further, a description of what Skemp means by reflective intelligence would be in order. He has illustrated the character of reflective intelligence by contrasting it with, and showing its relationship to, "sensori-motor intelligence."

The perception of numbers involves the awareness of properties such as those held in common ". . . by three eggs, in three cups, on three plates, with three spoons . . . ."[19] When the child can transcend properties most closely related to sensory stimuli and indicate his awareness of number properties independently of

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[18]R. R. Skemp, "Sensori-motor Intelligence and Reflective Intelligence," <u>Mathematics Teaching</u>, 22:17, Spring 1963.

[19]Skemp, "Reflective Intelligence and Mathematics," p. 46.

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configuration, he shows sensory intelligence. His awareness of relationships between sensory stimuli as distinct from resemblance between sensory stimuli is taken as a criterion of intelligence.[20] Skemp considers motor intelligence as ". . . awareness of relationships between actions, such as filling up and emptying out, putting together and taking apart, taking away and putting back."[21] A child manipulating objects and showing awareness of the relationships between, say, physical addition and subtraction manifests motor intelligence. Sensori-motor intelligence thus involves ". . . the perception of relationships between objects and groups of objects presented to the senses, and between the individual's own various motor activities."[22]

Building on these concepts, Skemp has defined reflective intelligence as:

> . . . the functioning of a second order system which:
>
> (i) can perceive and act on the concepts and operations of the sensori-motor system,
> (ii) can perceive relationships between these concepts and operations, and
> (iii) can act on them in ways which take account of these relationships and of other information from memory and from the external environment.[23]

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[20] Ibid., p. 47

[21] Ibid.

[22] Ibid., p. 49

[23] Ibid.

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Reflective thought is a second order system, aware of and acting on the mental representations of the sensori-motor system. In other words, reflective intelligence involves the ability of the mind to become aware of and to manipulate its own concepts.[24] Herein lies one of the important implications of the concept of reflective intelligence. If the concept is a valid one and if its relationship to mathematics learning can be demonstrated, any teaching method that increases a pupil's awareness of the concepts and operations he uses will help him to operate more effectively in mathematical situations. Requiring students ". . . to explain and describe what they do, and thereby directing their reflective awareness to their methods, will help their progress."[25]

An example of an act of reflective thought occurs in the approach to a problem that cannot be solved by routine application of already known methods but which requires new combinations or modifications of existing methods. This requires reflection on previously used methods, production of a modified sequence of

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[24] Ibid., p. 48; Skemp, "A Three-Part Theory for Learning Mathematics," p. 42.

[25] Skemp, "Reflective Intelligence and Mathematics," p. 51.

operations, and replacement of former sequences by the modified sequence. This requires knowing what has been done previously (possessing a system capable of being conscious of mental representations of one's operations) and being able to alter one's mental representations. An individual's reflection will be intelligent if and only if he can see the relationship between valid and invalid operations and the desired result. Otherwise he may be able to do similar tasks but his approach will be merely rote-learned.[26]

Not only has Skemp thus identified what he means by reflective intelligence, he has also developed measures of reflective activity with concepts and operations (as well as measures of concept and operation formation). These tests are designed to measure the extent to which a student is able to think reflectively. Skemp's tests of reflective intelligence played a central role in the study being reported.

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[26] Skemp, "The Psychology of Learning and Teaching Mathematics," p. 17.

III. OBJECTIVES OF THE STUDY

The purpose of the study being reported was to make a contribution to a developing theory of mathematics learning. The evidence available indicates that Skemp's theory is basically sound and that it will prove to be particularly fruitful in guiding research into the nature of mathematics learning. The study primarily consisted of an objective evaluation of the relevance of the concept of reflective intelligence to mathematics learning. However, a theoretical assessment of Skemp's theory has also been undertaken and has provided a basis for the formulation of tentative recommendations for teaching methods in mathematics education.

Specifically, the study was designed to gather objective evidence to indicate what the answers to the following three central questions might be.

1. Is the presence of reflective intelligence a factor of enough importance for success in mathematics that the addition of measures of reflective intelligence to measures of general intelligence significantly improves the prediction of one's performance in mathematics?

2. What effects on the relationships between measured reflective intelligence and mathematics performance result from taking into account student anxiety toward testing situations?

3. Are there any significant differences among the mean levels of reflective intelligence exhibited by boys or girls in age categories from ten to sixteen?

IV. DELIMITATIONS OF THE STUDY

The part of the study concerned with assessing the importance of reflective intelligence was confined to the eighth grade because the investigator was best prepared to assess mathematics performance at that level and because wide variation in manifested reflective intelligence was anticipated at that level on the basis of Piaget's description of the development of powers of formal reasoning.

The assessment of levels of reflective intelligence by grades was restricted to grades five through eleven because it was postulated that, if any marked change in measured reflective intelligence were observed, it would occur near the eighth grade level.

It was assumed that the test of mathematical understandings used is a valid measure of mathematics performance at the grade eight level.

The study was restricted to classes selected at random from the Edmonton Public School System.

V. MAJOR NULL HYPOTHESES

The research design was based on the following null hypotheses, which are here stated in general terms and

which correspond to the three central questions listed on pages 6 and 15.

Null Hypothesis 1. The efficiency of prediction of mathematics performance scores of grade eight students is not significantly improved by adding to a standardized battery of aptitude tests any of Skemp's reflective intelligence measures.

Null Hypothesis 2. The efficiency of prediction of mathematics performance scores of grade eight students is not significantly improved by adding to a battery of aptitude tests and reflective intelligence tests a measure of student anxiety toward test situations.

Null Hypothesis 3. For school children in grades five through eleven, there are no significant differences among the mean scores on Skemp's tests obtained by boys or girls in age categories from ten to sixteen.

VI. EXPERIMENTAL SETTING

After securing permission from the Associate Superintendent of Schools of the Edmonton Public School System to approach principals of schools within the system to seek their aid in executing the study, the investigator selected six classes of grade eight students and two classes at each of the grade five, six, seven, nine, ten, and eleven levels

from the appropriate populations of such classes in the Edmonton system by using a table of random numbers. Once the classes had been selected the principals were contacted. It was agreed that the investigator would administer tests to the grade eight classes during seven class periods at the rate of two a week in May and June, 1966. The grade eight students wrote a battery of ability tests, Skemp's tests, an anxiety test, and a test of mathematical understandings. The classes of grade five, six, seven, nine, ten, and eleven students were administered Skemp's tests in two class periods. The only difficulty encountered was that, by the time the grade ten and eleven classes could be worked into the investigator's testing schedule, a conflict with school final exams arose and it was decided that testing of these classes should be postponed to September, 1966, at which time it was carried out.

The grade eight students had been studying a "traditional" mathematics program which covered whole numbers, fractions, percent, graphs, plane figures, mathematics of the home, and mathematics of business. The grade five, six and seven students were studying a more contemporary mathematics program, and the grade nine, ten and eleven students were following the programs sequent to the grade eight traditional course.

VII. DEFINITIONS OF TERMS

In general, the meaning of each of the following terms and abbreviated forms of reference is indicated where it is first used in the report, but, for easy reference, the following list of definitions is presented at this point.

"Accommodation" refers to the process by which a schema is altered through experience or instruction, or both, to cope with new situations that are not completely assimilable to existing schemata.

"Anxiety" is defined for the purposes of this study as the experiencing of test situations as unpleasant and the experiencing of undue concern about success in school, about being wrong in test-like situations, about being promoted, and about not understanding what is expected.

"AR" refers to the _Abstract Reasoning_ test of the DAT battery.

"Assimilation" is defined as the process by which a problematic situation is handled by some schema which contains generally represented behaviour appropriate to the situation.

"CEEB" refers to the _College Entrance Examination Board_ of the United States.

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"DAT" refers to the Differential Aptitude Tests battery.

"Development of knowledge" is spontaneous, tied to development of the nervous system and of mental functions. It determines the totalility of mental possibilities and impossibilities at each stage of an individual's maturation.

"Equilibration" refers to the process of self-regulation by which one's cognitive structures react to compensate for external disturbances so as to move towards a state of equilibrium.

"Formulation" of a concept refers to the process by which the concept is made an object of consideration in itself, apart from the members of the class from which it was abstracted.

"G.C.E." refers to the general certificate of education exams written by fifth form students in England.

"Learning" involves the formation of cognitive structures in response to situations designed to provoke new schemata. Learning occurs as a function of total development and environmental influences.

"Motor intelligence" is defined as awareness of relationships among physical actions.

"NA" refers to the _Numerical Ability_ test of the DAT battery.

"Primary concepts" are defined as those concepts which are derived from direct sensory experience.

"Reflective intelligence" is defined as the functioning of a second order mental system which can perceive relationships among and act on the concepts and operations of the sensori-motor system, taking into account their relationships and information from the memory and external environment. In short, it is the ability of the mind to become aware of and to manipulate its own concepts and operations.

"SAT" refers to the CEEB _Scholastic Aptitude Test_.

"SCAT" refers to the _Cooperative School and College Ability Tests_, _School Ability Test_, _Form 4A_.

"Schema" is defined as a cognitive structure which has reference to a class of similar action sequences from past experience. It is the structure common to all those actions that an individual considers to be equivalent.

"Schematic learning" is defined as the kind of learning which makes use of an organized structure of knowledge (a schema) and builds more knowledge on to this structure.

"Secondary concepts" are defined as those concepts that are derived from other concepts.

"Sensori-motor intelligence" is defined as the perception of relationships between objects and groups of objects presented to the senses and between one's own actions with objects.

"Sensory intelligence" is defined as the ability to transcend properties most closely related to sensory stimuli to perceive relationships among objects as distinct from resemblances.

"SK4(1)" refers to Skemp's Concept Formation test which is given in preparation for the Reflective Action with Concepts test.

"SK4(2)" refers to Skemp's Reflective Action with Concepts test.

"SK6(1)" refers to Skemp's test of Operations Formation which is given in preparation for the Reflective Action with Operations test.

"SK6(2)" refers to Skemp's Reflective Action with Operations test.

"SMU-II" refers to the 1966 version of the Special Mathematical Understandings test devised by the investigator.

"SR" refers to the Space Relations test of the DAT battery.

"S-R" is used as an abbreviation for stimulus-response.

"TAS" refers to Sarason's Test Anxiety Scale, which has been used with university students.

"TASC" refers to Sarason's Test Anxiety Scale for Children and to the scale in its modified form as it was used in the study being reported.

"VR" refers to the Verbal Reasoning test of the DAT battery.

VIII. OUTLINE OF THE REPORT

The present chapter is an introduction to and a description of the nature of the study. Chapter II consists of a review of the literature that gives a theoretical framework for the study and that reports relevant research. Chapter III includes a detailed description of the design of the experiment and of the statistical procedures used in analyzing the data. The results from the statistical analyses are presented in Chapter IV. Chapter V is devoted to a summary of the investigation along with conclusions, limitations of the study, implications for mathematics education, and implications for further research.

CHAPTER II

THEORETICAL FRAMEWORK FOR THE STUDY AND RELATED RESEARCH

I. INTRODUCTION

Skemp was led to formulate his theory because he felt that existing theories of learning failed to explain adequately the nature of mathematical learning and thinking. To illustrate the nature of his departure from traditional approaches and to facilitate assessment of his theory in terms of other currently influential psychologically oriented approaches to the problem of mathematics learning, the first part of the present chapter is devoted to reviewing the educational contributions made in this century by traditional learning theories and to reviewing contemporary psychologically based theories that seem particularly relevant to considerations of the nature of mathematics learning. The reviews are written from the point of view of a mathematics teacher who is primarily interested in whether or not a given theory has practical implications for classroom teaching and learning situations. The second part of the chapter is devoted to a review of research

studies considered to be particularly relevant to the investigation presently being reported.

In the present chapter the discussions have been restricted primarily to considerations of the works of writers who have been (or promise to be) most influential in the development of psychologically based learning theories. For this reason no discussion of the contributions of Robert Davis, for instance, is included because his "discovery teaching" work has been pursued along practical, sub-theoretical lines.

II. LEARNING THEORY AND EDUCATION--AN OVERVIEW

In the period since the turn of the century, the dominant themes in the development of theories of learning in North America have come from the connectionist-behaviourist and from the Gestaltist-field theorist schools of thought. What seems paradoxical to an educator is that, though it has been somewhat sterile in terms of educational implications relevant to situations involving higher mental processes, the behaviourist tradition has flourished on this continent and research is still being vigorously pursued within its frame of reference. On the other hand, while the points of view of Gestaltists and field theorists have had considerable influence on educational practice during the past few decades, basic research

within these frameworks has waned on the North American continent. This state of affairs might very well be attributable to the "scientific" respectability of behaviourist research techniques, which focus on observable quantifiable activities of learners, as contrasted with the somewhat intuitive, qualitative assumptions and hypotheses characteristic of Gestalt and field theory.

In the early 1900's the dominant figure on the American psychological scene was Thorndike who, with his emphasis on objectivity and the development of generalizations by induction from data, developed model techniques for the application of scientific methods to educational problems. Thorndike's conception of learning as a matter of establishing connections blended nicely with the doctrine of "social utility" prevailing in the early part of the century because social utility could be used as a criterion for deciding what connections should be encouraged in the learner's behaviour. Thorndike's laws of learning, buttressed by a mass of scientific evidence, furnished educators with two obvious general rules:
"(1) Put together what should go together and keep apart what should not go together. (2) Reward desirable connections and make undesirable connections produce

discomfort."[1] According to Thorndike, educational methodology should be designed to foster comparisons and contrasts so that connections might be established between responses and situations. Adaptation to novel situations and purposive behaviour he felt should be treated as special cases of the effects of previously established connections. Quite naturally his system came to be called <u>connectionism</u>. It stressed the distinction between stimulus and response and emphasized that making a connection (a desirable, satisfying bond) between stimulus and response is the central psychological event in the teaching-learning process. Thorndike's wide appeal can be largely attributed to the scientific, quantitative nature of his system, to his concern with the everyday problems encountered in classroom learning situations, and to the enormous amounts of data he collected to support his views.[2]

Contemporaneous with Thorndike's promotion of connectionism was Watson's popularization of "behaviourism"

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[1]Frederick J. McDonald, "The Influence of Learning Theories on Education (1900-1950)," <u>Theories of Learning and Instruction</u>, Ernest R. Hilgard, editor (The Sixty-third Yearbook of the National Society for the Study of Education, Part 1. Chicago: University of Chicago Press, 1964), p. 8.

[2]<u>Ibid</u>., pp. 5-11.

--the study of behaviour alone without regard for conscious experience. By "behaviour" Watson meant nothing more than muscle movement. He regarded all learning as classical conditioning. To explain learning in terms of conditioned reflexes, he postulated the "principle of frequency," which states that the more frequently one has made a given response to a given stimulus, the more likely will be the occurrence of that response to that stimulus again. His "principle of recency" asserts that the more recently one has made a given response to a given stimulus the more likely that response will be made to that stimulus again. Watson and his successor, Guthrie, both maintained that stimulus-response bonds are strengthened simply by the response occurring in the presence of the stimulus. Psychologists holding this point of view have been called contiguity theorists, whereas subscribers to Thorndike's point of view have been referred to as reinforcement theorists because of their contention that the "effect" of the stimulus-response bond is the strengthening agent.[3]

Behaviourism and pure Thorndikean connectionism were soon to fall into disfavour in educational circles. Contributing to this decline was Dewey's attack on the

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[3]Winfred F. Hill, Learning: A Survey of Psychological Interpretations (San Francisco: Chandler Publishing Company, 1963), pp. 33, 35, 37, 58.

concept of the reflex arc. Dewey viewed stimulus and response as being organically related, not sharply distinguished. "'Mediated experiences,' events in relation to their adjustive function . . ."[4] he considered to be the central psychological events in learning activities. What the learner is already doing when a stimulus is given determines what his perception of the stimulus will be just as truly as the stimulus affects his subsequent actions. According to Dewey, stimulus and response are related by a circuit rather than by an arc or broken segment of a circle. The basic motivational construct in Dewey's system was "interest," in the sense of being absorbed in, being engaged in, or caring about. Interest of the learner in the learning task was considered to be the critical link between where the learner is and where the teacher would like him to be. Another major conception in Dewey's system was that of the role of "aims" in action. He viewed intelligent behaviour as involving foresight about the results of an act and the use of this foresight to control ordering, selection, and observation. This view contrasted sharply with Thorndike's notion that the teacher is to decide which connections are to be established,

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[4]McDonald, op. cit., p. 11.

putting a premium on skills and habits and coming near to denying purposive behaviour. Dewey's perception of total coordination in the learner's activities anticipated the Gestaltist point of view.[5]

In the late 1920's and early 1930's a growing dissatisfaction among psychologists and educators regarding the failure of behaviourism and connectionism to account for purposive behaviour produced a favourable climate for the introduction of Gestalt psychology to America. Gestalt psychology, with its new insight doctrine, was highly compatible with Dewey's description of the individual's capacity for setting and solving his own problems.[6] Following the lead of Wertheimer, the founder of Gestalt psychology, Köhler and Koffka vigorously criticized trial-and-error learning as conceived by Thorndike and the behaviourists. The Gestaltists persistently denounced the atomistic, mechanistic features of stimulus-response associationism and pointed to its failure to introduce organizational or structural concepts.[7]

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[5] Ibid., pp. 10-13. [6] Ibid., p. 19.

[7] Ernest R. Hilgard, "The Place of Gestalt Psychology and Field Theories in Contemporary Learning Theory," Theories of Learning and Instruction, Ernest R. Hilgard, editor (The Sixty-third Yearbook of the National Society for the Study of Education, Part 1. Chicago: University of Chicago Press, 1964), p. 73.

Köhler's experiments, which demonstrated that apes could learn to obtain rewards without laboriously stamping out incorrect responses and stamping in new ones, highlighted the notion of insightful learning as an alternative to trial and error. Insight had never been abandoned as a concept by the layman, but American psychologists, under Thorndikean and Watsonian influence, had generally come to view the learner as reacting somewhat stupidly to the pushes and pulls of the environment. Educators, with their notions of freeing intelligence for creative activity, received the new learning theories based on Gestalt views most enthusiastically.[8]

Gestalt psychology was very successful in the field of perception, demonstrating the effects of background and organization on phenomenally perceived processes. The theory that a percept consists of elements of sensation bound together by association was successfully attacked. According to Gestaltists, perception should rather be regarded in terms of a "whole," a Gestalt. The arguments used against sensationlike elements in perception were turned against the reflex arc in learning. Koffka's

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[8]Ernest R. Hilgard and Gordon H. Bower, Theories of Learning (third edition; New York: Appleton-Century-Crofts, 1966), pp. 229-231.

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treatment of learning proceeded under the assumption that the laws of organization of perception are applicable to learning. The guiding principle in learning was considered to be that psychological organization tends toward a state exhibiting such properties as regularity, simplicity, and stability. One illustration of this guiding principle was the "law of closure" which states that the direction of behaviour is toward an end situation bringing closure. "In a problematic situation the whole is seen as incomplete and a tension is set up toward completion."[9] The role of past experience in affecting present processes was dealt with in terms of a theory of memory traces which assumed that a trace persists from prior experience to represent the past in the present and that the present process can select and reactivate a memory trace structurally related to the ongoing process, resulting in recall or recognition. Insightful learning was associated with a restructuring of parts of a perceived field to emphasize certain relationships and to facilitate generalization of previous learning to cope with subsequent experiences. Insight occurs when the various components of a situation are grouped in consideration of their relation to the whole situation.[10]

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[9] Ibid., p. 235.

[10] Ibid., pp. 233-246.

Gestalt psychology and Lewin's "field theory" have many points in common, and each of the approaches is often closely identified with the other. The "field" in field theory is used in a sense analagous to the scientific connotation that regards "field" as a definite frame of reference marking the limits of interaction of the components of an event.[11] Lewin was most active in social psychology and his work continues to be very influential in that area. In the one article he wrote that dealt specifically with learning theory, Lewin asserted that the field which influences an individual's behaviour should be described in the way in which it is perceived by the individual at the time of the behaviour rather than in "objective physicalistic" terms. The teacher must understand the psychological world in which the child lives. It is wrong to substitute the teacher's world for that of the individual. Furthermore, Lewin postulated that efficient analysis of a situation derives from considering the situation first as a whole and then examining the parts and their interrelationships in the

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[11] George W. Hartmann, "The Field Theory of Learning and its Educational Consequences," The Psychology of Learning, Nelson B. Henry, editor (The Forty-first Yearbook of the National Society for the Study of Education, Part II. Chicago: University of Chicago Press, 1942), p.172.

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context of the whole. Lewin conceived of learning as consisting of changes in cognitive structure such as differentiation among the parts of a perceived field, perception of new patterns, and organization of seemingly unstructured components of a situation into orderly structures.[12]

From the early 1930's to the late 1950's, E.C. Tolman was very active on the psychological scene. He borrowed ideas from both the behaviourist and field theorist traditions. The theory Tolman developed was first called purposive behaviourism and then sign-gestalt theory. It emphasized the cognitive nature of learning but rejected introspection and conscious experience--his theory was concerned only with objective behaviour. Tolman emphasized the relation of goals to behaviour, and he analyzed behaviour in terms of fairly large common sense units (the molar approach) rather than in terms of atomistic muscle movements (the molecular approach). He postulated intervening variables to account for the individual's cognitions, his perceptions and

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[12]Kurt Lewin, "Field Theory and Learning," The Psychology of Learning, Nelson B. Henry, editor (The Forty-first Yearbook of the National Society for the Study of Education, Part II. Chicago: University of Chicago Press, 1942), pp. 216-219, 223-228.

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beliefs about the world, and the effect of such cognitions, perceptions, and beliefs on goal directed behaviour. Tolman held that, in learning, an individual forms a <u>cognitive map</u> in the sense of knowledge of what-leads-to-what and a <u>sign-gestalt expectation</u> in the sense of an expectation that the world is organized in certain ways.[13]

The present status of Gestalt, field, and sign-gestalt theories appears to be a matter of opinion. According to Hilgard,[14] by the 1950's interest in Gestalt psychology had largely subsided in America. Furthermore, he felt that Lewin's views on learning had not been sufficiently influential in the early 1960's to justify retaining a chapter on field theory in the third edition of <u>Theories of Learning</u>.[15] It is Hilgard's opinion that the influence of the Gestaltists, particularly in the field of problem solving and creative thinking, has been accepted but transformed rather than neglected. A consensus of opinions reported by Hilgard appears to hold that Gestalt psychology and cognitive learning theories of the

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[13]Hill, <u>op</u>. <u>cit</u>., pp. 114-122.

[14]Hilgard, <u>op</u>. <u>cit</u>., p. 57.

[15]Hilgard and Bower, <u>op</u>. <u>cit</u>., pp. v-vi.

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sort espoused by Lewin and Tolman have died of success by having their basic ideas and concerns assimilated into what is Psychology. On the other hand, according to Kohler, the logic of behaviourism is so ascendant in North America that even problems of perceptual contours are now being considered by American psychologists in atomistic terms. He does not believe that Gestalt's teachings have been assimilated but rather that its basic ideas have been continually rejected.[16] In a view somewhat contrary to opinions that field theory is dormant, an educational psychologist has identified two of the leading contenders on the 1966 learning theory scene to be reinforcement and conditioning, as exemplified by B. F. Skinner's "operant conditioning," and cognitive-field theory, as first advanced by Lewin but refined by contemporary psychologists such as Bruner.[17] Bruner's views, which are particularly relevant to the framework from which Skemp's theory is derived, will be reported in a later section. A discussion of Skinner's views is given in the following brief description of contemporary developments within stimulus-response theory. The

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[16] Hilgard, op. cit., pp. 55-62, 72-73; Hilgard and Bower, op. cit., pp. 261-263.

[17] M. L. Bigge, "Theories of Learning," National Education Association Journal, 55:18-19, March, 1966.

following paragraphs round out the present overview of relations that have evolved in this century between learning theory and education.

According to W. F. Hill,[18] most learning theories of academic psychology have a stimulus-response orientation. That the S-R approach has continued to flourish is attributable largely to its acceptance of cognitive and purposive interpretations from the Gestaltists and field theorists. Two of the prime interests in S-R theory today are cognition-oriented mediating responses and purpose oriented drive and reinforcement notions. Three major stimulus-response approaches, associated with Hull, Skinner, and Guthrie, exemplify general behaviourist theory as it has been pursued over the past quarter century in experimental psychology.[19]

The work of Hull has been continued, since his death in the early 1950's, by his student K. W. Spence. A description of Spence's system will serve to show the direction in which the Hullian tradition has moved. Spence,

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[18]Winfred F. Hill, "Contemporary Developments within Stimulus-Response Learning Theory," Theories of Learning and Instruction, Ernest R. Hilgard, editor (The Sixty-third Yearbook of the National Society for the Study of Education, Part 1. Chicago: University of Chicago Press, 1964), p. 27.

[19]*Ibid.*, pp. 27-28.

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like Hull, postulates that independent (stimulus) and dependent (response) variables are linked by a number of intervening variables such as: excitatory potential, the strength of the tendency to give a particular response to a particular stimulus; habit strength, a reflection of permanent learning; and drive, a motivational variable. According to Spence, the level of habit strength depends exclusively on the number of times the response has occurred to the stimulus. Incentive motivation is explained in terms of "fractional anticipatory responses," which are portions of a total response conditioned to the cues of a given goal (e.g., salivation and chewing motions in response to a picture of a steak). The fractional anticipatory responses produce stimuli which have motivational effects.[20]

In contrast, Skinner rejects the concept of intervening variables and seeks to establish laws of behaviour relating only stimulus and response. Skinner is interested primarily in predicting the behaviour of single individuals rather than in making statistical predictions for groups. He believes that:

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[20] *Ibid.*, pp. 31-37.

> Given a powerful independent variable, a reliable dependent variable, and adequate experimental controls, behavior is lawful enough to permit accurate prediction of what an individual organism will do under a given set of conditions.[21]

Skinner's main contribution to learning theory has been his distinction between respondent and operant behaviour. According to his view, respondent behaviour is elicited reflexively by particular stimuli whereas operant behaviour is emitted by the organism without being evoked by any particular, indentifiable external stimulus. Skinner treats these two kinds of behaviour within the frameworks of classical and operant conditioning theories, respectively. Skinner and his followers have also classified and analyzed reinforcement schedules, interpreted avoidance learning, analyzed verbal behaviour, and interpreted social phenomena in terms of learning principles. No attempt has been made to explain why reinforcers are reinforcing.[22] Skinner's formulation continues to generate a considerable volume of experimental research analyzing operant behaviour. Most programmed learning sequences follow Skinner's approach to learning by breaking learning tasks into very small steps and by giving the learner

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[21]*Ibid.*, p. 37. [22]*Ibid.*, pp. 37-40.

continual positive reinforcement through immediate feedback and minimization of errors.[23]

The third mainstream in recent S-R approaches to learning is that associated with Guthrie. In 1952, Guthrie reaffirmed his conviction that the basic law of learning is association by contiguity. He supported his contention by reference to animal experiments. More recently, he has placed greater emphasis on the notion of modification of stimulus reception by changes in the orientation of the receptor. He has expanded this concept to include scanning, the purposeful discovery of a stimulus by systematic variation in receptor orientation. Virginia Voeks and W. K. Estes, in attempting to give Guthrie's ideas a rigorous, deductive form, have given considerable impetus to the builders of statistical models for learning theory. However, Guthrie's few direct disciples appear to have become primarily concerned with laboratory based animal experiments.[24]

The preceding descriptions support the view that at least some S-R theorists have attempted to come to

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[23]John B. Murray, "Review of Learning Theories," <u>The Catholic Educational Review</u>, 63:21-22, January, 1965.

[24]Hill, "Contemporary Developments within Stimulus-Response Learning Theory," pp. 40-46.

grips with cognitive notions by introducing such concepts as mediating responses that produce stimuli for further responses. However, even leading S-R theorists admit that their learning theories have little present relation to practical educational issues. For instance, Spence maintains that learning theorists can best achieve their goals by studying artifically simple situations, which makes their results inapplicable to complex educational problems. R. M. Gagné, who has found it useful to concentrate on taking into account the hierarchical nature of the component operations making up learning tasks, has found suggestions in the reinforcement or "distribution of practice" vein to be of little assistance in the development of training methods. Skinner, on the other hand, has seen and initiated wide application of his brand of S-R theory in the field of programmed learning, promoting the view that efficient learning requires a response to be quickly reinforced and that desired behaviour can be "shaped" by successive approximations.[25]

Throughout the recent literature one finds recurring criticisms of the behaviourist framework by educators

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[25]Winfred F. Hill, "What Can the Psychology of Learning Offer Education?" Journal of Teacher Education, 14:444-447, December, 1963.

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and educational psychologists who indicate that they are anything but satisfied with recent developments. For example, J. B. Biggs[26] contends that, while behaviourist observations with respect to learning contain a grain of truth, such observations relate only to the promotion of rote or unstructured learning, a minor concern in education. He cites Skinner's conditioning of "superstitious" behaviour in pigeons as an illustration of the fact that such training is alogical because the connection between stimulus and response is based only on the extrinsic reward rather than on any logical relation between S and R. In a similar vein, M. Mayer has written:

> From the common-sense standpoint, all behavioristic theories suffer one impossible weakness. _The learning which these theories explain bears no necessary logical relation to the realities in the environment_. It is always possible that the reward has come for reasons that have nothing to do with the response. The animal never understands the nature of the problem, never knows _why_ the "operant" operates and, therefore, all his learning is potentially _superstitious_.[27]

W. F. Hill has noted that, early in this century, Thorndike, Watson, and Wertheimer, though far apart on many

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[26]John B. Biggs, "Towards a Psychology of Educative Learning," _International Review of Education_, 11:77, 1965.

[27]Martin Mayer, _The Schools_ (New York: Harper and Brothers, 1961), p. 75.

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issues, nevertheless considered implications for education as among the most important aspects of their work. They closely related the study of learning in the laboratory with that in the classroom. Subsequently, however, learning psychologists have generally concentrated on building precise, formal, and theoretical models of laboratory learning to the neglect of educational considerations. What educators have been able to gain from psychological learning theory, in the absence of extensive exchange of ideas, has been primarily in the form of hints and suggestions as to what variables might be important in educational contexts.[28] Furthermore, learning theory is by nature descriptive and, as such, is not necessarily a basis for prescribing how to teach. For example, the mere fact that learning can be described as taking place in small increments built up by a process of reinforcement does not necessarily imply that this is the best way to present learning tasks. Leaping from description to prescription requires an act of faith not necessarily based on factors relevant to instruction.[29]

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[28] Hill, "What Can The Psychology of Learning Offer Education?" p. 443.

[29] James B. Macdonald, "Myths About Instruction," Educational Leadership, 22:573, May, 1965.

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In consideration of the foregoing review, it is not surprising that the history of liaison between educators and learning theorists has been described not as a joyous and fruitful union but as a fitful affaire involving ambivalence and rejection.[30] However, not to end on a completely negative note and to summarize developments described in the preceding overview, the writer presents a paraphrase of thirteen "principles," potentially useful for practice, that Hilgard and Bower[31] have distilled from learning theories. The first seven principles are emphasized in S-R theory, and the remaining ones are emphasized in cognitively oriented theories.

1. The learner should be active.
2. Frequency of repetition is important in acquiring skill.
3. Reinforcement is important.
4. Practice in varied contexts promotes generalization and discrimination.
5. Novel behaviour can be enhanced through cueing, "shaping," and imitation of models.
6. Drive and motivational conditions are important in learning.
7. Conflicts and frustrations arising in the learning of difficult discriminations need to be recognized and their resolution or accommodation provided for.
8. The perceptual features in the display of a problem to a learner are important conditions of learning.

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[30]Biggs, op. cit., p. 77.

[31]Hilgard and Bower, op. cit., pp. 562-564.

9. The organization of knowledge should be an essential concern of teachers and educational planners (e.g., progression from simplified wholes to more complex wholes).
10. Learning with understanding is more permanent and more transferable than rote learning.
11. Cognitive feedback allows the learner to accept or reject a provisional trial on the basis of its consequences.
12. Goal-setting by the learner is important in motivating his learning.
13. In addition to the convergent thinking that leads to logically correct solutions, divergent and creative thinking should be nurtured.

The preceding overview of the psychological-educational climate in North America has been presented to place in perspective consideration of educational implications from the work of Piaget, Bruner, Dienes, and Skemp, whose psychologically oriented theories are having considerable impact on contemporary approaches to the teaching of mathematics. On the following pages the highlights of the points of view developed by these men are reviewed, interrelated, and discussed in terms of implications for the teaching of secondary school mathematics.

III. CONTEMPORARY PSYCHOLOGICALLY ORIENTED THEORIES RELEVENT TO MATHEMATICS EDUCATION

Piaget's Cognitive Theory of Intellectual Development

Reflecting upon the educational implications that can be gleaned from traditional learning theories, one is likely to reach Skemp's conclusion that theories based on such notions as conditioning, reinforcement learning, sign learning, and perceptual learning are not adequate for the classroom. Results from experiments involving simple mental processes at the sub-language level have not been satisfactory in terms of educational implications because classroom learning processes are complex in nature and they are carried on at the language level.

> A theory is required which takes account (among other things) of the systematic development of an organised body of knowledge, which not only integrates what has been learnt, but is a major factor in new learning . . .[32]

Such a theory has been put forward by Piaget, a psychologist whose prodigious and insightful work has been extremely influential in contemporary attempts to improve the learning and teaching of mathematics and of virtually

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[32] R. R. Skemp, "The Need for a Schematic Learning Theory," British Journal of Educational Psychology, 32:133, 1962.

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every school subject. A basic notion in the theory derived by Piaget and his Geneva group is that of a schema, a cognitive structure which has reference to a class of similar action sequences from past experience. A schema can be defined as the structure common to all those acts that an individual considers to be equivalent. Any problematic situation requiring behaviour which is already generally represented in cognitive structure by a schema is handled by being assimilated to the schema. If the individual has no completely relevant schema, new behaviour sequences are built by experimentation or instruction, or both, to enable existing schemata to accommodate to the new situation. Adaptation of an individual to his environment results from the interplay of assimilation and accommodation.[33]

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[33]John H. Flavell, The Developmental Psychology of Jean Piaget (New York: D. Van Nostrand Company, Inc., 1963), pp. 52-53; Bärbel Inhelder, "Some Aspects of Piaget's Genetic Approach to Cognition," Thought in the Young Child, William Kessen and Clementina Kuhlman, editors (Monographs of the Society for Research in Child Development, Vol. 27, No. 2, Serial 83, 1962), p. 25; Oscar G. Mink, "Experience and Cognitive Structure," Piaget Rediscovered, R. E. Ripple and V. N. Rockcastle, editors (Ithaca, N. Y.: School of Education, Cornell University, 1964), p. 68; E. A. Lunzer, Recent Studies in Britain Based on the Work of Jean Piaget (Occasional Publication No. 4. London: National Foundation for Educational Research in England and Wales, 1960), pp. 46-47.

Paradoxically, even though an early formulation of this potentially useful point of view was available in English in 1950,[34] the standard text for graduate work in learning theory, Hilgard's Theories of Learning,[35] makes no reference to it, and, in fact, does not even include Piaget's name in the list of authors referred to. Skemp has hypothesized that Hilgard may not have considered Piaget's Psychology of Intelligence, the 1950 formulation referred to above, applicable to learning theory discussions in view of the fact that the author of Theories of Learning has asserted that problems of the nature and measurement of intelligence are outside the scope of his text. Since it would appear safe to assume that Hilgard's text represents and largely determines the state of learning theory in North America (considering its almost universal use in graduate level courses on learning), the lack of applications of learning theory to education is understandable, in view of Hilgard's neglect of the one theory that seems most promising for education.[36] Furthermore, in the 1966 third edition of

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[34]Jean Piaget, The Psychology of Intelligence (London: Routledge and Kegan Paul, 1950). English Edition.

[35]Ernest R. Hilgard, Theories of Learning (second edition; New York: Appleton-Century-Crofts, 1956).

[36]Skemp, "The Need for a Schematic Learning Theory," p. 134.

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Theories of Learning,[37] though a further elaboration of Piaget's theory became available in English in 1958[38] and subsequently in several secondary sources, reference to Piaget's work is relegated to a footnote[39] in which it is explained that the meaning of Piaget for learning theory has not been well enough worked out for exposition.

Among the more succinct and yet lucid descriptions of Piaget's cognitive theory of intellectual development are those given by Piaget[40] himself in addresses to conferences in the United States in 1964 and by his close associate, Bärbel Inhelder,[41] at a 1960 American conference. These sources have guided the organization of and have contributed most to the review of Piaget's theory that is presented in succeeding paragraphs.

Piaget makes a distinction between the *problem of development* and the *problem of learning*. According to Piaget, the development of knowledge is spontaneous, tied

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[37] Hilgard and Bower, *op. cit.*

[38] Bärbel Inhelder and Jean Piaget, *The Growth of Logical Thinking from Childhood to Adolescence* (New York: Basic Books, Inc., 1958). English Edition.

[39] Hilgard and Bower, *op. cit.*, p. 571.

[40] Jean Piaget, "Development and Learning," *Piaget Rediscovered*, R. E. Ripple and V. N. Rockcastle, editors (Ithaca, N.Y.: School of Education, Cornell University, 1964), pp. 7-19.

[41] Inhelder, *op. cit.*, pp. 19-34, 34-40 (Discussion).

to the development of the nervous system and of mental functions, aspects of embryogenesis. Development of children ends only in adulthood. Development determines a certain totality of structures of knowledge (a totality of possibilities and impossibilities) for each person at each stage of growth. Learning, on the other hand, is provoked by situations (for example, situations created by a teacher), as opposed to being spontaneous, and is limited to a single problem or a single structure at any moment. A particular social environment is indispensable for the realization of an individual's mental possibilities, and such realization can be accelerated or retarded by the nature of cultural and educational conditions. Piaget believes that each element of learning occurs as a function of total development, contrary to the atomistic view held by some psychologists that development is the cumulation of a series of specific learned items.[42]

The idea central to the development of knowledge is that of an <u>operation</u>. An operation is an <u>interiorized</u> action which can modify an object of knowledge. For example, an operation could consist of constructing a classification (joining objects in a class), of putting things in a series, or of counting or measuring.

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[42]Piaget, "Development and Learning," pp. 7-8; Inhelder and Piaget, <u>op</u>. <u>cit</u>., p. 337.

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> To know an object is to act on it. To know is to modify, to transform the object, and to understand the process of this transformation, and as a consequence to understand the way the object is constructed.[43]

An operation is reversible in the sense that it can take place in both directions (e.g., joining and separating). The attainment of reversibility requires more than an ability to actually undo an observed transformation, which capability Piaget calls renversabilité, in that the individual must anticipate in thought a return to the state prior to the transformation. This anticipation, with its implied annihilation of two inverse relationships, characterizes the attainment of reversibility.[44] Never isolated, an operation is always part of a total structure and is linked to other operations. A logical class exists in the total structure of classification; an asymmetrical relation exists in the context of seriation, the natural, basic operational structure; a number exists in the series of numbers, which constitute a structure.[45]

In the child's development of operational structures, the basis of knowledge, Piaget has distinguished four main

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[43] Piaget, "Development and Learning," p. 8.

[44] Inhelder, op. cit., p. 35.

[45] Piaget, "Development and Learning," pp. 8-9.

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stages: a sensory-motor, pre-verbal stage, extending through approximately the first eighteen months of life; a pre-operational stage extending from about eighteen months to about seven years; a concrete operations stage from about seven years to about eleven or twelve years; and a formal operations stage which begins at about eleven or twelve years of age. Although the order of succession of stages is constant, the chronological ages corresponding to the stages vary a great deal from culture to culture and individual to individual.[46]

During the sensory-motor stage is developed the practical knowledge on which later representational knowledge is built. For example, the schema of a permanent object is constructed so that, towards the end of this stage, the infant will try to find a previously seen object when it is outside his perceptual field whereas he would not have done so in the first few months. Consequently, a notion of practical or sensory-motor space is constructed along with notions of temporal succession and elementary sensory-motor causality.[47]

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[46] _Ibid_., pp. 9-10; Flavell, _op_. _cit_., p. 86; Inhelder, _op_. _cit_., pp. 26-27.

[47] Piaget, "Development and Learning," p. 9.

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The pre-operational stage is marked by the beginnings of language, of the symbolic function, and, consequently, of thought or representation. Reconstruction of all that was developed on the sensory-motor level must occur at the level of representational thought. Throughout this stage of pre-operational representation, there is no evidence of conservation, the psychological criterion of the attainment of reversible operations. For example, given two equal balls of plasticene and asked to roll one of them into a sausage shape, the child will assert that there is more or less substance in the sausage than in the ball, depending on whether he focuses on the increase in length or on the decrease in diameter. He cannot seem to relate different aspects or dimensions to one another, and he tends to be deceived by his perceptions. He cannot mentally imagine the sausage being re-rolled into a ball of the original size. His concrete thought processes are irreversible.[48]

In the stage of concrete operations extending from about age seven to about age eleven, a thought structure, not yet separated from its concrete context, is formed. The first operations appear along with systems of operations that can be carried out simultaneously--in contrast

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[48]Ibid.; Inhelder, op. cit., pp. 25-26.

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with sensory-motor actions which are carried out only in succession. The operations are concrete in the sense that they operate on real objects. Example of operations developed in this stage are those of classification, of ordering, and of construction of the notion of number, as well as operations of spatial and temporal nature, operations of the elementary logic of classes and relations, and operations of elementary mathematics, geometry, and physics. The yet incomplete systems of operation are characterized by two forms of reversibility: negation, in which a perceived change is seen to be cancelled by its corresponding negative thought operation; and reciprocity, in which, for example, "being a foreigner" is seen as a reciprocal relationship and before-behind spatial relationships are seen as relative. At this level, the two forms of reversibility are employed independently of one another. The operations of this level form a system having the structure of a mathematical group with a method for producing a composition of elements (the operations can be combined), with an identity element, with an inverse for every element, and with the associative property of composition. The transformation from the pre-operational stage to the stage of concrete operations is made when the child's mental acts become organized into such group-

like structures rather than being isolated and unrelated as they were formerly.[49]

The fourth stage, that of formal operations, beginning on the average at about eleven or twelve years of age, is characterized by the development of formal, abstract thought operations with which the child, now virtually an adolescent, can reason in terms of hypotheses and not only in terms of objects. A stable system of thought structures is formed at about fourteen or fifteen years of age in a rich cultural environment (such as that in Geneva). The adolescent constructs new operations of propositional logic to supplement the operations of classes, relations, and numbers. He attains new thought structures that, as his experimental procedures become more and more effective, come to approximate a lattice structure when he is using methods of combinatorial or formal logic and the structure of a group of four transformations when he is using methods involving proportionality. (It is not the case that Piaget started with the notion that the thought processes *should* conform to the laws which govern logical and mathematical structures,

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[49]Piaget, "Development and Learning," p. 9; Inhelder, *op. cit.*, p. 26.

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but rather that the facts gleaned from over twenty years of research have shown that cognitive development approximates, but never completely attains, these structures.) In any event, the adolescent's thought structure is marked by a higher degree of reversibility than in previous stages. Negation and reciprocity become united in a completely operational system. The adolescent can identify all possible factors relevant to a problem under investigation, and he can form all possible combinations of these factors, one at a time, two at a time, three at a time, and so on. He can form hypotheses, construct experiments to test the hypotheses against reality, and draw conclusions from his findings. He need no longer confine his attention to what is real but can consider hypotheses that may or may not be true and work out what would follow if they were true. That is to say, in addition to considering what _is_ he can consider what might be. The hypothetico-deductive procedures of mathematics and science have become open to him.[50]

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[50]Piaget, "Development and Learning," pp. 9-10; Inhelder, _op. cit._, pp. 22, 27-28; Irving Adler, "Mental Growth and the Art of Teaching," _The Arithmetic Teacher_, 13:579, November, 1966; D. E. Berlyne, "Recent Developments in Piaget's Work," _British Journal of Educational Psychology_, 27:8-10, February, 1957.

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An example of formal operational thought is that carried on by the adolescent in coping with a problem in which he is given five bottles of colourless liquid, of which the first, third, and fifth combine to form a brownish colour, the second is neutral, and the fourth bleaches out colour. The problem is to find out how to produce the coloured solution. The adolescent gradually discovers the combinatorial method, reasoning through the construction of a table of all possible combinations and experimentally determining the effect of each factor. This type of reasoning is beyond younger children. Similarly, given a series of mounted rings of different diameters, a projection screen, a candle, and a series of equally spaced positions in which to place the rings between candle and screen, an adolescent, asked to place all the rings so that a single unbroken shadow of a "ring" is produced on the screen, will arrive at the conclusion that the proportion between the size of ring and the distance from the candle must be kept constant.[51]

According to Piaget, the development from one set of mental structures to another is explained by the operation of four factors: maturation, experience, social transmission,

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[51]Inhelder, op. cit., p. 27.

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and equilibration. He states that none of these is sufficient by itself to account for the preceding descriptions of mental development, but he considers the fourth, equilibration or self-regulation, to be the fundamental factor.[52]

Even though maturation of the nervous system plays an indispensable role in development, it does not explain everything because the average age at which each of the various stages occur (but not the order of occurrence) varies widely from society to society.[53]

Experience of objects, of physical reality, is also a basic factor in the development of cognitive structure, but it does not explain everything. For example, some of the concepts which appear at the stage of concrete operations cannot be drawn from experience alone. Consider the fact that a child becomes cognizant of conservation of substance at approximately age eight, but he does not assert that weight or volume is conserved until some time later. Weight and volume can be perceived directly, but how can the amount of substance be considered without notions of weight and volume? The child comes to

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[52] Piaget, "Development and Learning," p. 10.

[53] Ibid.

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understand that when there is a transformation of the shape of a quantity of plasticene, for instance, something must be conserved because the transformation can be reversed so that the plasticene can be returned to its original condition. Since it is not yet the weight and not yet the volume that is seen to be conserved, the notion of conservation of substance is simply a logical necessity--no experience can show the child at this level that there is the same amount of substance. Furthermore, experience is of two psychologically distinct kinds: <u>physical experience</u> and <u>logical-mathematical experience</u>. Physical experience conforms to our usual notions of acting on objects and gaining some knowledge about the objects through the process of abstraction. Logical-mathematical experience, on the other hand, is drawn from the <u>actions</u> effected on the objects. For example, a child discovers that, no matter how he arranges a certain set of pebbles and no matter in what direction he counts them, he always has the same number. To make a sum and to order the pebbles, action is necessary. The child has discovered that the action of putting together (summing) is independent of the action of ordering--this is a property of the actions, not of the pebbles. Herein lie the beginnings of mathematical deduction, which are further developed by

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the interiorization of the actions so that they can be combined without the need of pebbles. Before the formal operations stage, the coordination of such actions requires the support of concrete material, but it later leads to logical-mathematical structures in which operations are combined through the use of symbols and earlier logical-mathematical structures are used as a point of departure in thinking about new combinations. The source of logic lies in the coordination of such actions as joining together and ordering. Logical-mathematical experience, an experience of the individual's actions, not an experience of the objects themselves, is necessary before there can be operations.[54]

Social transmission, linguistic or educational, is a third basic factor. As an example of societal effects on development, consider Piaget's observation that the emergence of formal thinking corresponds to the age at which society expects the child to begin assuming adult roles. Not the onset of puberty, but the pressure to assume adult roles is the distinctive feature of adolescence in modern civilizations.[55] However, in order for a

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[54]*Ibid.*, pp. 11-13.

[55]Inhelder and Piaget, *op. cit.*, pp. 335-336.

child to receive information from society he must have a structure that enables him to assimilate the information. Consequently, social transmission by itself is not adequate to explain development. Ordinarily a young child cannot be taught higher mathematics because he does not yet have the structures that would enable him to understand. Similarly, young children do not have a sufficiently well developed linguistic structure to understand that "some of" implies inclusion of a subclass in a class.[56]

Equilibration, the fourth factor, serves to relate the other three factors. An individual engaged in the act of knowing is led to react to compensate for external disturbances so that a state of equilibrium can be reached. The process of equilibration leads to operational reversibility, which is characterized by an equilibrated system in which a transformation in one direction is compensated for by a transformation in the other direction. This active process of self-regulation embodies the concept of feedback from the individual's interactions with his environment, and it takes the form of a succession of levels of equilibrium. A system is in equilibrium when a disturbance which modifies the state of the system has its counterpart in a spontaneous action which compensates for it.

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[56]Piaget, "Development and Learning," p. 13.

Levels of equilibrium can be identified according to the probability of the occurrence of various possible forms of compensation. Laws of equilibrium determine, at each stage of development, the best forms of adaptation compatible with maturation, experience, and influence of the social milieu. For instance, the pre-operational child can only cope with one dimension at a time and is led to assert non-conservation of a substance whose perceived form is altered, whereas a child in the concrete operations stage is able to take account of compensating changes in dimension (focusing on the transformation and not on the final configuration) to arrive at the notion of conservation.[57]

In reviewing The Growth of Logical Thinking from Childhood to Adolescence, Bruner[58] questioned the usefulness of Piaget's notion of equilibrium, maintaining that the concept of reversibility of operations is sufficient, and that only confusion is added by introducing the equilibration construct. Piaget has responded to this criticism by stating that reversibility is a logical idea while

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[57]Ibid., pp. 13-14; Inhelder and Piaget, op. cit., pp. 245-246.

[58]Jerome S. Bruner, "Inhelder and Piaget's The Growth of Logical Thinking: I, A Psychologist's Viewpoint," British Journal of Psychology, 50:365, 368-369, 1959.

equilibrium is a causal idea which permits the explanation of changes in thinking forms by means of probabilistic schema. In the formation of the notion of conservation, the following stages of strategy are distinguished: (1) considering one dimension to the neglect of others is the most probable strategy in the beginning, (2) emphasizing the second dimension becomes the most likely as a result of employment of the first strategy, and (3) oscillating between observed compensating changes in the different dimensions becomes most likely as a result of the preceding strategies. Accordingly, the process of equilibration is characterized by sequential control with increasing probabilities, furnishing a beginning for causal explanations of reversibility rather than duplicating the idea. The process of equilibration starts at the level of self-regulation and sensory-motor feedback and leads to operational reversibility and intelligent thought at higher levels of development. Every new problem produces a disequilibrium which is recognizable by the dominant types of errors made in coping with the situation. A solution is often arrived at by synthesizing a new operation from formerly distinct operations to produce a new state of equilibrium. Consider, for example, the derivation of the concept of ordinal number from the process of cardination

and the action of ordering. The "second" element in a row is the one that has one predecessor, and the "third" element is the one with two predecessors, and so on.[59]

So far Piaget's view of the development of cognitive processes has been presented, but what can be said about his view of the learning process? He maintains that the learning of logical-mathematical structure can be accomplished only if the teacher can build the structure to be learned from simpler, more elementary logical structures. This view is derived from the notion that logical structures are not the result of direct physical experience. They can be grasped only through the function of internal equilibration or self-regulation in coping with the characteristics of various actions. By way of example, Piaget and Inhelder have led five-year-olds to grasp conservation of number by getting them to drop beads simultaneously into a glass they can see and one they cannot. Usually children cannot grasp conservation of number until seven or eight if they are presented situations in which a one-to-one correspondence between two rows of objects is set up and then one row is spread out. The former (hidden glass) structure is

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[59]Jean Piaget, "The Genetic Approach to the Psychology of Thought," Journal of Educational Psychology, 52:279-281, December, 1961.

analagous to the latter, but it is embodied in a simpler situation. It has been found that children can generalize from the simpler situation to grasp the concept in the more difficult setting. Learning of a complex structure is possible if such learning is based on natural development from and perception of relationships to more simple structures. In short, the learning of structures seems to obey laws similar to those governing the natural development of these structures. Learning is subordinated to development. Learning is only effective if it is lasting, if it can be generalized to new situations, and if the learner's operational level is raised. Naturally developed cognitive structures satisfy these criteria, and "learned" structures should satisfy nothing less. Furthermore, learning is possible only when there is active assimilation on the part of the learner, assimilation in the sense of integration of reality into cognitive structures.[60]

Having thus far reported descriptions of Piaget's theory, it would perhaps be in order to contrast the theory briefly with some aspects of behaviourist and Gestaltist theories. Piaget considers the classical stimulus-response framework as incapable of explaining

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[60]Piaget, "Development and Learning," pp. 15-18.

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cognitive learning. He believes that a stimulus is significant only if it can be assimilated by a cognitive structure, and that it is the structure that produces the response. Between the stimulus and the response is an organism with its structures. The response exists in the structure before the stimulus is perceived. These thoughts have led Piaget to propose that the stimulus-response relation should be considered circular--in the form of a schema which is not simply one way.[61] This is vividly reminiscent of Dewey's point of view.

A psychologist who tried to translate the findings of Piaget's Geneva group into Hullian language found it necessary to introduce _transformation_ _response_ and _internal reinforcement_ constructs, which Piaget considers to be what he has called _operation_ and _equilibration_, respectively. With these additions Hull's conceptualization is seriously modified and is no longer characterized by the simple association of stimulus response theory.[62]

On the other hand, Piaget has found several points of agreement between his theory and classical Gestalt theory. For instance, both theories reject the attribution

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[61] _Ibid._, p. 15.

[62] _Ibid._, pp. 18-19.

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of any special faculty to intelligence, agree that cognitive activities and the realities on which they act are structured totalities, and consider intelligence and perception as systems in progressive equilibration. A major difference between the two theories is the fact that the Piagetian schema is conceived of as a more dynamic and modifiable structure than the Gestalt is. Whereas the Gestalt is considered as an expression of a certain level of neural maturation given a certain perceptual field and is not seen as the product of past interactions with the environment, the Piagetian schema is always the product of differentiation, generalization, and integration of earlier schemata resulting from attempts to accommodate to the environment.[63]

More important, however, in the present context is a consideration of what implications for education can be drawn from Piaget's theory. His theory has a great many implications that have a bearing on attempts to improve learning efficiency and the art of teaching. In the succeeding paragraphs is reported a sampling of such implications that can be found in the literature and that

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[63]Flavell, op. cit., pp. 72-73.

seem to be relevant to secondary school mathematics teaching and learning.

A recent article by Irving Adler[64] includes the following implications from Piaget's theory. The mathematical experiences a child is given at any age should be experiences he is ready for in terms of the stage of mental growth he has reached, and they should help prepare him to advance to the next stage. Before introducing a child to a new concept, one should test him to see if he has the prerequisites for forming the concept, and, if not, he should be provided with appropriate developmental experiences. Especially in the lower grades, concepts should be built from appropriate concrete experiences. To help a child overcome his errors in thinking, provide him with experiences that will expose the errors, thus assisting the process of accommodation which will lead him to cope adequately with the situation at hand. Flexible thinking is based on reversible operations. By analogy, it would seem beneficial to teach operations in inverse pairs and to stress their relationship (e.g., the relation between addition and subtraction). Children in the stage of concrete operations can be helped to gain

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[64]Adler, op. cit., pp. 581-584.

a better grasp of relations among subsets of a set if they are given experiences in manipulating sets of objects to explore relationships among sets, subsets, intersection, union, and hierarchical inclusion. Combinatorial analysis is based on the formation of Cartesian products of sets. Children can be readily taught systematic ways of forming these products by using tree diagrams and rectangular arrays. Since mental growth is encouraged if one is given opportunities to see things from many points of view, teaching should give children the opportunity to use a wide variety of approaches in tackling problems. For example, in teaching geometry, not only the traditional synthetic approach should be used, but approaches using analytic techniques, vectors, and isometries of the plane could also be included. As mental growth is associated with discovery of invariants, a systematic search for the features of a situation that remain unchanged under a group of transformations should aid in developing awareness of and understanding of the relationships involved in the situation. Since the onset of formal thinking occurs at about age eleven or twelve, it would appear psychologically sound to introduce short units of deductive reasoning from hypotheses as early as the sixth grade.

An interesting and insightful further observation made by Adler is that the concrete operations used by an individual are "concrete" in the sense that they are mental operations involving some real system of objects and relations perceived by the person. What is "concrete" is relative to the person's past experience and mental maturity. While the kindergarten child considers the union of two beads with three beads as a concrete operation but the addition of 2 and 3 as not, the introductory algebra student considers 2 + 3 as concrete but not x + y. The student of introductory abstract algebra considers the additive group of integers as concrete but does not consider the concept of an abstract group to be concrete. So the progression goes, and it is evident that "concrete" operations are used not only in the concrete operations stage, in which they are the most advanced operations of which the child is capable, but also at all succeeding levels of learning. In the development of new concepts at any level it is necessary to proceed from the concrete to the abstract.[65] Piaget's notion of a "vertical _décalage_" is consonant with this point of view. Vertical _décalage_ refers to a formal similarity between the structure of

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[65] _Ibid._, p. 584.

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thinking at one level of development and that at a higher level of functioning. For example, there are structural similarities buried in the toddler's construction of a mental map of his immediate surroundings and the elementary school child's map-making project.[66]

Further in the area of educational implications, Piaget has stated that an individual's apparent failure to grasp the most basic concepts of elementary mathematics stems not from a lack of any special aptitude but rather from affective blocking or inadequate preparation. Furthermore, the frequent failure of formal education can be traced to the fact that it begins with language, illustrations, and narrated action rather than real, practical action. Preparation for mathematics education should begin in the home with the encouragement of concrete manipulations that foster awareness of basic logical, numerical, and mensurational relationships. This practical activity should be systematically developed and amplified throughout the primary grades until it takes the form of elementary physical and mechanical experiments by the time secondary education begins.[67] In a similar vein, Eleanor Duckworth has said:

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[66]Flavell, op. cit., pp. 22-23.

[67]Jean Piaget, "The Right to Education in the Modern World," Freedom and Culture, UNESCO (London: Wingate, 1951), pp. 95-98.

> You cannot further understanding in a child simply by talking to him. Good pedagogy must involve presenting the child with situations in which he himself experiments, in the broadest sense of that term--trying things out to see what happens, manipulating things, manipulating symbols, posing questions, and seeking his own answers, reconciling what he finds at one time with what he finds at another, comparing his findings with those of other children.[68]

In much the same line of thought, Piaget has been cited by Duckworth as having taken issue with those who recommend that children be taught the "structure" of a subject area, that is to say, the unifying themes of the subject area, so that they will be able to relate individual aspects to the general structure. During a discussion period at a conference, Piaget said:

> The question comes up whether to teach the structure or to present the child with situations where he is active and creates the structures himself. . . .The goal in education is not to increase the amount of knowledge, but to create the possibilities for a child to invent and discover himself. . . .Teaching means creating situations where structures can be discovered; it does not mean transmitting structures which may be assimilated at nothing other than a verbal level.[69]

In another context, Piaget has warned against the danger often inherent in school learning of leading the

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[68]Eleanor Duckworth, "Piaget Rediscovered," The Arithmetic Teacher, 11:497, November, 1964.

[69]Ibid., p. 498, citing verbal comment made by Piaget.

child to false accommodation to words or to authority rather than to reality as it presents itself. It is preferable for a teacher not to correct a child's schemata, but to provide situations that will lead the child to correct them himself.[70]

In discussing teacher education, Piaget has been reported as saying that even adults can learn better by <u>doing</u> than by being told about such things as how to teach effectively. Furthermore, he is of the opinion that prospective teachers should have the opportunity to question children in a one-to-one situation so that they will realize how difficult it is to make oneself understood. Teachers in training should also pursue an original investigation to determine what children actually think about some problem. In endeavouring to communicate individually with a number of children in this way, the prospective teacher may be able to overcome the illusion that he can talk successfully to a whole class of children at once.[71]

In line with the view that intellectual development brings a gradual transformation of overt actions into mental operations, a key concept in Piaget's theory,

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[70] <u>Ibid</u>., citing verbal comment by Piaget.

[71] <u>Ibid</u>., citing verbal remarks by Piaget.

a teacher would do well to assist the internalization and schematization process by having students perform actions with less and less direct support from external entities. For example, the child might be led to operate directly on physical objects, then on pictorial representations, then on cognitive anticipations of operations not actually being performed, and so on, until the original external operations take place internally and independently of the environment. In addition, since social interaction is essential in developing the multiperspective view essential for rationality and objectivity, group activities in the form of projects and discussions should be encouraged.

To round out the present consideration of educational implications from Piaget's theory, the writer submits a quotation of a rather thought-provoking point of view expressed in the introduction to the report of the Cambridge Conference on School Mathematics, a report which calls for a major reorganization of the grade school mathematics curriculum and for compression of high school and college mathematics offerings.

> We made no attempt to take account of recent researches in cognitive psychology. It has been argued by Piaget and others that certain ideas

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[72] Flavell, op. cit., pp. 368-369.

> and degrees of abstraction cannot be learned until certain ages. We regard this question as open, partly because there are cognitive psychologists on both sides of it, and partly because the investigations of Piaget, taken at face value, do not justify any conclusion relevant to our task. The point is that Piaget is not a teacher but an observer--he has tried to find out what it is that children understand, at a given age, when they have been taught in conventional ways. The essence of our enterprise is to alter the data which have formed, so far, the basis of his research. If teaching furnishes experiences which few children now have, then in the future such observers as Piaget may observe quite different things. We therefore believe that no predictions, either positive or negative, are justified, and that the only way to find out when and how various things can be taught is to try various ways of teaching them.[73]

Well, such is the view of a number of mathematicians and applied mathematicians. In essence, one has the feeling that Piaget would not disagree, especially if due consideration of the implications from his theory for effective teaching methods were taken into account. In fact, both Bruner and Dienes, who are strongly influenced by Piaget's work, have developed teaching procedures which would tend to support the view that mathematical abstractions can be taught to children at a very young age indeed. It is to descriptions of the work of Bruner and Dienes that the present review now turns.

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[73]Educational Services Incorporated, Goals for School Mathematics, The Report of the Cambridge Conference on School Mathematics (Boston: Houghton Mifflin, 1963), pp. 3-4.

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Bruner's Theories of Thinking and of Cognitive Growth

Investigations by Bruner and his associates into the thinking processes of adults resulted in the publication in 1956 of A Study of Thinking.[74] This early work was concerned with the process of concept attainment--

> . . . the search for and testing of attributes that can be used to distinguish exemplars from non-exemplars of various categories, the search for good and valid anticipatory cues. . . . [as distinguished from] concept formation--the inventive act by which classes are constructed.[75]

On the assumption ". . . that virtually all cognitive activity involves and is dependent on the process of categorizing,"[76] Bruner and his collaborators devised tasks to facilitate investigation of the strategies used by adults in attaining certain concepts. In one such task the subject was given an array of eighty-one cards exhibiting one, two, or three figures, the figures having the form of crosses, circles, or squares, having one, two, or three borders, and being coloured green, black, or red. The subject was told that a "conjunctive concept" meant a set of cards that share a certain set of attributes. Some practice

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[74]Jerome S. Bruner, Jacqueline J. Goodnow, and George A. Austin, A Study of Thinking (New York: John Wiley and Sons, 1956).

[75]Ibid., pp. 232-233.

[76]Ibid., p. 246.

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examples were given, and then the subject was told that the experimenter had a certain concept in mind, and a card illustrating the concept was pointed out. The subject was then asked to choose cards one at a time to which the experimenter would respond by indicating whether the card was an exemplar of the concept or not. He could make one hypothesis concerning the concept after each choice, and he was asked to try to arrive at the concept as efficiently as possible. Four different strategies were discerned: a _simultaneous-scanning_ strategy in which the subject uses the results from each choice to deduce which hypotheses are tenable and which are not; a _successive-scanning_ strategy in which a single hypothesis is tested with each choice and in which the subject limits his choices to instances providing a direct test of his hypothesis; a _conservative-focusing_ strategy in which the finding of a positive instance is used as a focus in making a sequence of choices each of which alters only one attribute of the focus card to see whether a positive or negative instance is generated; and a _focus-gambling_ strategy in which a positive instance is used as a focus, but more than one attribute is changed from choice to choice. Similar experiments were devised to investigate

the attainment of "disjunctive concepts," defined by cards exhibiting any one of a number of attributes, and "relational concepts," defined by cards exhibiting a specifiable relationship between the defining attributes.[77]

In Bruner's view, concept attainment tasks can be stripped down to six essentials: (1) An array of instances characterized by certain attributes is to be tested in order to attain a concept; (2) With each instance tested a person makes a tentative prediction or hypothesis about some combination of attributes that forms a concept which is an abstraction of his experience; (3) With succeeding instances a search for validation or refutation of the hypothesis is undertaken; (4) Each hypothesis and test yields potential information by limiting the number of attributes to be considered relevant; (5) The sequence of decisions based on hypotheses and tests forms a strategy for discovering valid cues (attaining the concept); and (6) Decisions made by the individual about the nature of the instances encountered are seen as having consequences for the decision maker (in terms of success of failure).[78]

One finding from Bruner's concept attainment studies was that it is possible to describe and evaluate strategies in a systematic way in terms of their objectives and

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[77] Ibid., pp. 41-43, 83-89. [78] Ibid., pp. 233-234.

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in terms of steps taken to achieve the objectives. For example, conservative focusing is a strategy which, when one is dealing with conjunctive concepts and is not being limited as to time or number of instances that can be tested, is found to be moderately efficient for accumulating information, cognitively unstrenuous, and almost failure proof. However, subjects tend to switch to the more risky but faster strategy of focus gambling if the number of testable instances is limited. Another finding was that the tendency to fall back on cues relevant in previous analagous situations may be helpful but may also form a major obstacle for adoption of efficient strategies to handle new situations. It was found that there was a general tendency for subjects to be unable or unwilling to use efficiently information based on negative instances or indirect tests of hypotheses. Another tendency was for subjects to prefer common-elements (conjunctive concept) strategies of cue-searching, even when not relevant. Disjunctive concepts were found more difficult to attain than conjunctive ones. It was also found that greater amounts of information were required before subjects would abandon hypotheses that fit their general notions.[79]

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[79] _Ibid._, pp. 235-238.

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Bruner has also investigated the advantages for thinking accruing from the formation of a "generic coding system" composed of conceptual categories into which environmental situations can be placed and from which relevant information about unobserved properties of the situation can be read off. In order to form a coding system, the learner must have experience with a variety of dissimilar instances of the concept as well as examples of other concepts. A major advantage of such coding systems is the extended use of concepts to organize information and to manipulate environmental facts. By regrouping or recoding the various events in the environment to facilitate ease of storage, handling, and retrieval, the mind is freed from the overwhelming burden of storing a massive array of unrelated items of information.[80]

Since 1956, Bruner has become increasingly interested in cognitive development. The unmistakable influence of Piaget on his work in this area is amply acknowledged by Bruner in the preface to one of his recent books.[81]

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[80] Jerome S. Bruner, "Going Beyond the Information Given," Contemporary Approaches to Cognition, A Symposium Held at the University of Colorado, Jerome S. Bruner and others (Cambridge: Harvard University Press, 1957), pp. 41-67.

[81] Jerome S. Bruner and others, Studies in Cognitive Growth (New York: John Wiley and Sons, 1966), pp. vii-xv.

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The theories of Bruner and Piaget do have points of disagreement, but these are minor in comparison to the overall fundamental agreement.

In Bruner's view, cognitive growth depends largely on the mastery of techniques transmitted by the culture. A prime example of such techniques is that of the use of language. To benefit from contact with recurrent regularities in the environment, one must represent them in some way. Retrieval and use of what is relevant from past experience depends upon how this experience is coded and processed. Bruner has postulated three modes of representation by which humans construct models of the world. The first, called enactive representation, summarizes events by means of appropriate motor responses (actions). The second, called iconic representation, ". . . summarizes events by the selective organization of percepts and of images, by the spatial, temporal, and qualitative structures of the perceptual field and their transformed images."[82] The third, called symbolic representation, represents objects and events by means of arbitrary symbols. An example of such a symbol is a word, which neither points

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[82]Jerome S. Bruner, "The Course of Cognitive Growth," American Psychologist, 19:2, January, 1964.

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directly to its referent nor resembles it as an image. The three modes of representation occur in the life of the child in the order given in the preceding sentences, each depending on the preceding one for its development yet all remaining more or less intact throughout life. Once the child has reached the symbolic level and has succeeded in internalizing language, he is enabled to represent and to transform the regularities of experience with greater flexibility and power than he could formerly. At the enactive level, actions cannot be transformed, and children thinking in terms of actions merely perform one action after another, perhaps sometimes reordering them. Images, the basis of iconic representation, can be transformed but they lack generality. Once the child has learned to use symbols, though, it may well be that he can use actions and images quite arbitrarily, like symbols, and, indeed, he may very well use all three modes of representation simultaneously.[83]

Among the educational implications from Bruner's theory is his postulation that if it is true that intellectual development follows the enactive, iconic, symbolic

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[83] Ibid., pp. 1-4; Z. P. Dienes, Mathematics in Primary Education (Hamburg: International Study Group for Mathematics Learning, 1966), p. 38.

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sequence, then it is likely that an optimum learning sequence will progress in the same way.[84] That is to say, learning cycles can be regarded as microscopic copies (microcosms) of the developmental cycle (the macrocosm). This view was originally expressed by Dienes as a possible way of interpreting Piaget's stages of growth.[85]

Further in the area of educational implications, Bruner has asserted that instruction should provide information to the learner about the higher order relevance of his efforts. Instruction is provisional in the sense that the object is to make the learner self-sufficient.[86]

Bruner has expressed another view about teaching that is very much in line with Skemp's opinion on the same topic.

> If learning or problem solving is proceeding in one mode--enactive, ikonic [sic], or symbolic--

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84Jerome S. Bruner, "Some Theorems on Instruction Illustrated with Reference to Mathematics," Theories of Learning and Instruction, Ernest R. Hilgard, editor (The Sixty-third Yearbook of the National Society for the Study of Education, Part 1. Chicago: University of Chicago Press, 1964), p. 313.

85Dienes, Mathematics in Primary Education, p. 39.

86Bruner, "Some Theorems on Instruction Illustrated with Reference to Mathematics," pp. 316, 318-319.

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> corrective information must be provided either in the same mode or in one that translates into it. . . . In mathematics particularly, one finds that teachers often provide information for correction in a highly symbolized notation when, in fact, the student is proceeding either without knowledge of the symbolic language used or by the use of some sort of approximate imagery. The result is incomprehension or defeat.[87]

Undoubtedly on the basis of his work on coding and representation, Bruner has stressed the importance of teaching so that the student grasps the structure of a subject. Teaching so that fundamental understanding is achieved of the underlying principles giving structure to a subject will help a student to generalize so that he can cope with new situations. It will also help the student to retain the essence of what he has learned because his learning is organized in terms of principles and ideas.[88] As noted previously, Piaget would want to qualify this point of view to ensure that the student actively develops his own notions of the structure of the subject rather than being "taught" the structure in meaningless isolation from what it organizes. That this view is in accord with Bruner's is brought out very well in Marilynne Adler's paraphrase of Bruner's views:

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[87]Ibid., p. 318.

[88]Jerome S. Bruner, The Process of Education (Cambridge: Harvard University Press, 1960), p. 31.

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> The most effective way to develop a complex coding system is to allow the child to _discover_ these basic concepts for himself. The most effective way to prevent true learning and growth is to present a "predigested" version of knowledge, a tidy catalogue of facts and figures.[89]

Perhaps the most well-known of Bruner's remarks is his contention that ". . . any subject can be taught effectively in some intellectually honest form to any child at any stage of development."[90] His basic assumption here is that it is only necessary to translate the structure of the subject into terms that are consonant with the child's characteristic way of looking at the world (which, in turn, is determined by his level of development). Bruner has asserted that readiness is only a half-truth because one can "teach" readiness by providing opportunities for its nurture rather than simply waiting for it to develop in due course. Readiness consists of mastery of the simpler skills on which more complex skills are based.[91] Piaget has been cited as saying that when

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[89]Marilynne J. Adler, "Some Educational Implications of the Theories of Jean Piaget and J. S. Bruner--Bruner," _Canadian Education and Research Digest_, 5:10, March, 1965.

[90]Bruner, _The Process of Education_, p. 33.

[91]Jerome S. Bruner, _Toward a Theory of Instruction_ (Cambridge: Harvard University Press, 1966), p. 29.

Bruner claims an individual can learn anything at any age, he overlooks the biological character of development.[92] Paradoxically, Bruner has marshalled Piaget's own theory in support of the view expressed at the beginning of this paragraph.[93] Building on this point of view and the one described in the preceding paragraph, Bruner has recommended that the curriculum be built around the great issues, principles, and values that have been discovered in each of the subject areas. He has urged further that these basic ideas be presented in a concrete form as early as possible and that they be re-presented and developed in more and more abstract forms as the child's thinking processes mature.[94] This concept of a "spiral curriculum" is quite in line with the view expressed by Piaget (and reported earlier in this chapter) about giving children early concrete experiences to lead gradually to more abstract ones.

Bruner, in collaboration with Dienes, has also carried out some experiments which successfully led young

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[92]"Psychology of Intelligence and Education," Pennsylvania School Journal, 114:318, March 1966, an abstract of an address by Piaget.

[93]Bruner, The Process of Education, pp. 34-38.

[94]Ibid., pp. 52-54.

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children to fairly sophisticated mathematical ideas by having them manipulate specially constructed concrete materials. Since the materials and method were Dienes', discussion of the generalizations arising out of such experiments is reported in the section on Dienes' work, which follows.

Dienes' Theory of Mathematical Learning

The development of Dienes' theory of mathematics learning has proceeded on the assumption of the correctness of the pioneering work of Piaget and associates. It has also been built on findings from a 1960-61 experimental collaboration between Dienes and Bruner at the Harvard Center for Cognitive Studies as well as on Dienes' prior theoretical work at Leicester University.[95]

In the late 1950's Dienes investigated the nature of children's thinking processes. His investigations were guided by notions of *constructive thinking* and *analytic thinking*. Constructive thinking involves the construction of classes out of seemingly unconnected events--putting things together to build a structure that will meet some set of requirements. Analytic thinking

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[95] Z. P. Dienes, *The Power of Mathematics* (London: Hutchinson Educational, 1964), p. 9.

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involves examination of constructively built structures to become fully aware of their properties with an aim of building more complex structures. This thinking about what has been built is a more sophisticated activity than the building. Analytic thinking is directed towards becoming aware of relationships between structures and parts of structures. Two findings from Dienes' investigations were that constructive thinking is the most dominant mode for young children, and that analytic thinking must be preceded by appropriate constructive activities if it is to be efficient.[96]

From a constructive point of view, the statement $\frac{1}{3} = \frac{2}{6}$ would involve visualization of a whole consisting of six objects arranged in three equal groups consisting of two objects each. From an analytic point of view, the same statement might be interpreted in terms of a symbolic relationship among the numbers 1, 3, 2, and 6 and identified as a ratio.[97] As another example of the contrast

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[96]Ibid., p. 26; Z. P. Dienes, An Experimental Study of Mathematics-Learning (London: Hutchinson, 1963), pp. 41, 61-63; Dienes, Mathematics in Primary Education, p. 52.

[97]Z. P. Dienes, "The Growth of Mathematical Concepts in Children through Experience," Educational Research, 2:10, 1959.

between constructive and analytic thinking, consider the fact that Dienes has found that a six-year-old very often will say that a long block made up of four cubes is three times as big as a single cube because in comparing them he mentally builds three more cubes onto the single cube to make the long block (constructive thinking). He does not appreciate the 1:4 relationship between the objects (analytic thinking). Dienes believes that such a child needs a great deal of constructive practice in making the long block from the cubes before he can achieve analytical insight.[98]

Proceeding from Piaget's description of intellectual growth and from his own observations of children, Dienes has identified three types of experience as essential in the development of a concept. The first is a "preliminary game," which consists of largely unstructured play with concrete material (manipulative play). The second is a "structured game" (rule-bound play), which consists of play directed by rules corresponding to the structure of a concept abstracted from the properties of the concrete material. The third is a "practice game," which serves to anchor the insights of the structured game in the child's

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[98]Ibid., p. 12.

experience. The practice game then serves as a preliminary game for the formation of a superordinate concept.[99] Proceeding in this frame of reference, Dienes has formulated three hypotheses to guide the development of appropriate pre-conceptual experiences that will fit children's natural concept formation cycles.

> Hypothesis A. Visual, tactile and muscular images must be formed to create perceptual equivalents of a concept. From the common essence of these will be abstracted the conceptual structure.
> . . . .
>
> Hypothesis B. The higher the level of generality at which a concept is formed, the wider its fields of possible application.
> . . . .
>
> Hypothesis C. A concept involving variables is best understood in its full generality if the variables contained in it are made to vary.[100]

Dienes has constructed ingenious forms of apparatus in accordance with these hypotheses to provide perceptual equivalents of mathematical concepts ranging from the most basic number concepts to those underlying quadratic functions, groups, and vector spaces. On the basis of results from experiments with children using these materials and being engaged in imaginative mental games, Dienes has organized his notions into three basic principles governing the formation of mathematical concepts:

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[99] Ibid., pp. 13-15.

[100] Ibid., pp. 16-17.

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> 1. <u>Dynamic Principle</u> [Revised]. . . . learning proceeds in cycles which follow one another in regular succession. Each cycle consists of approximately three stages: a preliminary and rather unstructured "play" stage, a more structured intermediate constructive stage, followed by the insight, then an anchoring stage in which the insight is more firmly fixed in place.
>
> . . . .
>
> The unstructured play . . . can be described . . . as manipulative play. This leads to the construction of more and more complex categories as the cycles progress; playing with each construction follows, and enables the player to discover the regularities. These regularities--the rule structure--once discovered are subjected to . . . analysis; this is the anchoring stage of the cycle and helps to fix the properties, possibilities and limits of the rule-structure in theory and in practice.[101]
>
> 2. <u>Mathematical Variability Principle</u>. Concepts involving variables should be learnt by experiences involving the largest possible number of variables. [All of which should be varied if full generality of the mathematical concept is to be achieved.]
>
> 3. <u>Perceptual Variability Principle.</u> To allow as much scope as possible for individual variations in concept-formation, as well as to induce children to gather the mathematical essence of an abstraction, the same conceptual structure should be presented in the form of as many perceptual equivalents as possible.[102]

While potentially useful, the "mathematical variability principle" has been found to produce problems in

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[101]Dienes, <u>An Experimental Study of Mathematics Learning</u>, pp. 156-157.

[102]Z. P. Dienes, <u>Building Up Mathematics</u> (London: Hutchinson Educational, 1960), p. 44.

application. Dienes has discovered that many subjects who can generalize two variables cannot generalize three. Perhaps with younger children it is preferable to generalize on several narrow fronts, abstracting to a broader front later.[103]

The "perceptual variability principle" has been referred to in other places as a "multiple embodiment principle."[104] It is central to applications of Dienes' theory in the sense that the presentation of many embodiments of a concept eventually leads to retention of only the essential mathematical structure which, in turn, can be applied to subsequent exemplars.

Constructive, "concretely" oriented activity is essential to provide the background for meaningful pursual of analytic, symbolically oriented learning activities. An example of constructive activity leading to analytic insight was that engaged in by a student taught the factorization of $x^2 + 2x + 1$ by Dienes. With other children, the student was asked to take a number of boxes and put in each box as many objects as he had taken boxes. Then

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[103] Dienes, <u>An Experimental Study of Mathematics Learning</u>, pp. 159-160.

[104] Dienes, <u>The Power of Mathematics</u>, p. 40.

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he was asked to take two more boxes and fill each of these with the same number of objects as in each of the boxes in the first set. A single object was placed on the table with the boxes. Somehow the students then learned, through considerable undirected manipulation, that it was possible always to place the objects into boxes so that the arrangement would consist of one more box than in the first set of boxes selected and in each box there would be one more object than there was originally. This activity gives the experimental background for the identity

$$x^2 + 2x + 1 = (x + 1)^2.$$

After this relationship had been learned from several embodiments, including the preceding one, some of the children began to wonder why it was always one more box and one more object per box. Suddenly, the student referred to above said:

> "Of course you empty one of these extra boxes into the first lot, putting one object in each. Then the extra object on the table will go in the remaining extra box into which we haven't yet put any extra objects. This is how you get one more box and one more object in each box."[105]

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[105] Z. P. Dienes, "Research in Progress," <u>New Approaches to Mathematics Teaching</u>, F. W. Land, editor (London: Macmillan and Company, 1963), p. 54.

He had reflected on what he had done and had ". . . become curious about the structure of events whose regularities he had gleaned from his experiences. This is an analytical insight, a doubling back upon our tracks."[106]

Consider how meaningful the symbolic representation of similar factorizations would be with this kind of concrete experience as a background. The following symbolic factoring procedure would follow quite naturally.

$$x^2 + 2x + 1$$

Separating the "two more" boxes gives

$$x^2 + x + x + 1.$$

Emptying the contents of one box into x of the boxes gives

$$x(x + 1) + x + 1.$$

Putting the extra object into the remaining box of the "two more" gives

$$x(x + 1) + (x + 1).$$

Uniting all the boxes gives

$$(x + 1) \text{ times } (x + 1).$$ [107]

It would be difficult to imagine a more natural approach than this one to a technique for factorization of trinomials based on the distributive property of multiplication over addition.

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[106] *Ibid.*, pp. 53-54. [107] *Ibid.*, p. 55.

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Dienes has made a useful distinction between abstraction and generalization. He considers abstraction to be a constructive activity in which a child forms a class of events by recognizing a property common to members of this class. On the other hand, generalization is considered to be a process of class extension--once a child forms a class he may realize that there is a more extensive class to which the properties of the original class apply (directly or by isomorphic analogy).[108]

Dienes' notions of manipulative play and rule-bound play correspond roughly to Bruner's enactive and iconic stages in the formation of a concept. Once children have reached a certain level of abstraction through manipulative and rule-bound play with various structured materials, it is found that they can manipulate symbols to use and to communicate to others the mathematical structures they have discovered. In fact, Dienes has related that one eight-year-old passed through the enactive, iconic, and symbolic stages associated with the

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[108] Z. P. Dienes, "On Abstraction and Generalization," Harvard Educational Review, 31:281-282, Summer, 1961; Dienes, Mathematics in Primary Education, p. 53.

development of notions of quadratic functions in less than five weeks.[109] Whereas Bruner has tended to emphasize the importance of symbols in the experiments carried out in collaboration with Dienes, the latter has emphasized the importance of the experiences leading to the formation of the concept. Bruner would contend that symbolic forms (such as word descriptions) are always at hand but merely need to be understood, while Dienes believes that order is created out of chaos through manipulation and then arbitrary symbols can be introduced to summarize what is known.[110]

Most of Dienes' observations are directly applicable to mathematics learning situations. However, perhaps a summary listing of implications for mathematics education would be in order. For instance, Dienes has found that, if approached in constructive modes, normal ten-year-olds can be led to understand the use of brackets, properties of squares (second degree powers), solution of linear and quadratic equations, and factorization of quadratic functions, to name only a few of the mathematical "abstractions" within their grasp. In fact, children can be given a large body of practical experiences throughout

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[109] Ibid.

[110] Ibid., p. 54.

elementary school so that a very extensive mathematics program can be undertaken in secondary schools.[111]

Dienes warns that when students learn to abstract a concept from only one type of embodiment there is danger of producing a perceptual block--i.e., such students may not be able to see that the principles or properties discovered are applicable in other situations. The implications from the multiple embodiment principle in this regard are evident.[112]

Throughout Dienes' writings there is the underlying assertion that concretely oriented activities are essential if analytic-symbolic learning is to be meaningful. Somewhat related to this basic view is the question raised by Skemp (and quoted on page 9 of this report) that pointed to an apparent contradiction between Dienes' claims of successful teaching of advanced mathematical topics (logarithms, indices, matrices, vectors, etc.) to pre-adolescents and Piaget's view that pre-adolescents are not ready for such abstract, formal, and logical ideas. Having raised the question, Skemp has described the concrete embodiments and methods used by Dienes in teaching

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[111]Dienes, "The Growth of Mathematical Concepts in Children through Experience," p. 28.

[112]Dienes, Mathematics in Primary Education, p. 53.

these concepts and has reached the conclusion that the mathematics is done by the deviser of the games played and not by the children. He has asserted that the children have not engaged in mathematical thinking of an analytic, reflective nature, but he has conceded that perhaps the activities would be of value in preparation for study of mathematics proper.[113]

At this point it is quite apparent that not only would Piaget's views not conflict with Dienes' claims but that both men would likely find points of agreement in Skemp's analysis. No contradiction exists in fact. Even if the children were to reach symbolic, analytic levels of thought about the concepts embodied in the games played (and Dienes has indicated that pre-adolescents do reach this level), the level of thinking attained could be attributed to the rich experiences that they would not normally have had.

With these remarks the focus of the present review turns from consideration of Dienes' contributions to a discussion of Skemp's theory in the light of what has been reported in this chapter so far.

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[113]R. R. Skemp, "The Psychology of Learning and Teaching Mathematics," Study No. 1 on the various aspects of the teaching of mathematics in secondary schools, Paris, UNESCO, 1962, pp. 15-16. (Mimeographed, Limited Distribution).

<u>Skemp's Three-part Theory of Mathematics Learning</u>

The three "parts," or the three main features, of Skemp's theory are: its elaboration of a method for the formation of mathematical concepts, its description of schematic learning as essential to mathematics learning, and its postulation that mathematics learning depends on the reflective use of intelligence. The reflective intelligence part of the theory has been discussed briefly in relation to the central questions of the present study on pages 7 to 14 of this report. However, the concept of reflective intelligence cannot be fully appreciated without some knowledge of the way in which it has developed. Accordingly, the present section is devoted to a description of the evolution of Skemp's theory and to a discussion of all three parts of the theory in terms of the way these are interrelated and in the light of the preceding reviews.

According to Skemp, mathematics learning involves processes at two levels: ". . . the 'invention of new means through mental combinations' . . . and the learning of the elements which are thus combined."[114] The former level, which is described in terms borrowed from Piaget,

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[114]R. R. Skemp, "An Investigation into Difficulties in Learning Mathematics by Children of Good General Intelligence" (unpublished Doctoral Dissertation, The University of Manchester, 1958), pp. 57-58.

includes the highest forms of creative thinking as well as the simple improvisations necessary for a child to solve a textbook problem that he perceives as new. The latter level includes the learning of the simple techniques from which more complex processes are formed. More specifically, what must be learned in arithmetic and all over again in mathematics are: (1) the basic facts such as 2 + 3 = 5, (2) the basic processes such as addition and multiplication, (3) the use of the basic facts and processes for routine tasks such as solving simple equations, (4) the use of known facts and processes in new combinations to cope with new tasks (problem solving), and (5) the discovery of new relationships and invention of new processes.[115]

Having thus described the various stages of mathematics learning, Skemp has examined how far classical conditioning and reinforcement theories can take one in accounting for such learning. Even at the first stage, learning the basic facts in beginning arithmetic, there are at least two kinds of learning: ". . . that aimed at laying a foundation of understanding, based on experience; and that aimed at learning all the basic facts about sums and products . . . ."[116]

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[115]Ibid.

[116]Ibid., p. 62.

If the first is neglected in favour of the second so that everything about sums and products is learned by rote, children may learn to do mechanical arithmetic but they may never form the concepts and operations which must be generalized to make the transition from arithmetic to higher levels of mathematics. Since classical conditioning is based on the premise that, before an S-R bond can be conditioned, the response must first occur "unconditionally" to the stimulus and since there is no unconditional stimulus to which the response is, say, "24," the linking of "24" with "3 times 8" cannot be explained. Consequently, conditioning cannot even account for the rote learning aspect of the first stage of mathematics learning. On the other hand, rote learning does resemble reinforcement learning of the Hullian type in terms of the gradual learning curve, the increased probability of recurrence of rewarded S-R bonds, the necessity for correct responses to <u>happen</u> before they can be reinforced, and the absence of understanding or insight. If one can successfully determine what event causes the reinforcement, learning by drill can be described in terms of reinforcement theory. However, the theory would have to be extended to take account of such psychological needs as teacher approval, avoidance of criticism, and desire to do well, which Skemp

considers to be the principal reinforcing agents in classroom learning by drill. But how can learning with understanding be explained? Perhaps Tolman's view that a set of cognitions rather than a motor response is what is reinforced comes closer to explaining such learning. According to Tolman, the results from one's various activities are taken to be reinforcing if they confirm the learner's expectations. The reinforcement of the response "15" to the stimulus "7 plus 8" may be by confirmation of the expectation that the teacher will respond affirmatively, that the teacher will respond with praise, or that when seven objects are joined with eight objects the result will be a group of fifteen objects. The third case is an example of cognitive learning with understanding whereas the second is one of rote learning, and the first could be an example of either, depending on whether the affirmation is interpreted as information or as a value judgement. The foundations of mathematical understanding are built on the gradual formation of concepts whose properties and relationships are derived from concrete experiences. If a child learns by drill without confirmation in terms of activities with real objects, "being right" comes to mean "being approved" or "avoiding criticism," and the basic mathematical concepts are never formed.[117]

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[117]Ibid., pp. 61-69.

While the learning theories cited can be interpreted so as to account for some of the features of arithmetic learning, a much clearer and more fruitful conception of the process can be obtained from Piaget's frame of reference:

> If a child has learnt by experience that three and five make eight, whatever the object, then he is in a position to _understand_ that three hundred plus five hundred equals eight hundred, without ever counting out those many beads or matchsticks. Which is to say, that the second fact is _assimilated_ to the existing schema "three plus five equals eight," the latter itself being derived from numerous repetitions of sensori-motor activities with three and five objects.
>
> This schema is an abstraction of that which is common to all operations of this kind; that which is invariant with respect to changes of the objects themselves.[118]

Pursuing this line of thinking, Skemp considers _understanding_ to mean assimilation to existing schemata, as distinct from _verbal learning_ (accommodation to words or authority) that remains isolated. It is important to form concepts before, or while, learning their verbal labels.[119]

Still considering the development of arithmetic ideas, learning the basic processes (the second stage) involves learning to use the basic number facts in the context of addition, subtraction, multiplication, and

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[118] _Ibid._, p. 70.

[119] _Ibid._, pp. 72-73.

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division of larger numbers and then of fractions and decimals. All that has been said about rote learning and understanding in the first stage can be applied at this stage. Symbols and mental manipulations are built on the schemata of the first stage. Since direct access to confirmation through concrete experience is no longer always available, it becomes particularly important that affirmation or contradiction by the teacher be interpreted in cognitive terms. An answer marked wrong should mean "not true" rather than "bad boy." That the basic arithmetic processes can be easily rote learned is a real danger because such processes will not readily be generalized in the transition to mathematics learning. To counteract over-dependence on rote learned procedures, students can be encouraged to check results to see if they are consistent with other mathematical knowledge. Invoking the criterion of self-consistency is a fruitful cognitive activity that can be used to bring out relationships such as those between reverse processes.[120]

The third stage, application of basic facts and processes to routine tasks, causes little difficulty if the formal processes have been mastered. Models can be

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[120]*Ibid.*, pp. 73-77.

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provided to help the pupil to organize his knowledge about the basic processes so he can cope with the tasks set. Several processes may be required in the solution of one task, so the student must learn to handle these simultaneously. Furthermore, he will have to increase his ability to abstract numerical information from other details.[121]

It is at the fourth stage, in which new combinations of known facts and processes are required for problem solving, that a pupil may begin to encounter difficulty. If understanding has accompanied drill up to this point, the necessary conceptual apparatus will have been built on a foundation of such understanding. However, if only the easier method of rote learning has been relied upon by the student, he will be unable to solve problems efficiently. He will continue to seek models for every problem encountered, but this will no longer do as new problems will arise that require intelligent modifications and combinations of existing concepts and processes, which is only possible if these have been understood.[122]

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[121] _Ibid_., p. 77.

[122] _Ibid_., pp. 72-78.

Skemp's primary concern has been with the four stages of learning so far discussed. The fifth stage, the discovery of new relationships and the invention of new processes, follows if the preceding four stages are well handled.

The foregoing analysis of arithmetic learning can be applied to mathematics learning. Considering the transition from arithmetic to algebra as an example, one finds that the development closely parallels the transition from sensori-motor activities to formal arithmetic.

> Just as the number concepts and arithmetical processes are abstractions from and generalizations of experiences with material objects, so are the algebraical concepts generalizations of experiences with numbers and arithmetical processes.[123]

Just as "3" is the property common to all groups of three objects, "the counting number x" represents any of the arithmetical concepts exemplified by 1, 2, 3, ... However, the superordinate concepts cannot be formed if the subordinate ones are not understood. In any case, it is clear that, with appropriate changes, all that has been said about arithmetic applies to mathematics but at a higher level of mental functioning. The learning of basic facts and processes can be by rote or by generalization from previous experiences, and so on.[124]

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[123]Ibid., p. 78. [124]Ibid., pp. 78-79.

So far, the thinking which led Skemp to develop his theory has been described. In order to minimize repetition, further discussion of Skemp's views on mathematics learning is deferred to the description of the theory itself, which follows.

Concept Formation. One of the simplest kinds of concept is a "class-concept," a set of properties which characterizes a particular class of objects. (This is the sense in which Bruner uses the term concept.) For example, "oak" is the name of a class-concept which represents the set of properties held in common by all oak trees. "Oak" is subordinate to "tree" which, in turn, is subordinate to "plant," and so on, forming a hierarchy of class-concepts. One of the easiest ways to communicate a class-concept is to cite a number of examples of objects which belong to the class. The more the examples given and the greater their variety, the more likely that one will be able to abstract the properties that define the concept. It is important to note that this way of communicating a concept makes use of concepts of lower order than the concept being demonstrated. There is no advantage in using examples of the same or higher order if the learner is not likely to understand them.[125]

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[125]Skemp, "The Psychology of Learning and Teaching Mathematics," pp. 6-7.

In teaching mathematics one has to communicate to students concepts of a relatively high order. Some of the concepts used in mathematics are: relational concepts, operations, abstractions, and generalizations.

> Examples of relational concepts are "equal," "greater than," "tends to," "function of," "isomorphic with." Examples of operations are "differentiation," "addition," "projection," "substitution." Examples of abstracting are "concept formation," "getting the equation," and every case in which arithmetical or mathematical data are derived from a practical problem. Examples of generalizing are the processes of developing ideas about fractional and negative indices starting from positive whole number indices, and about trigonometrical ratios of angles of any magnitude from the definitions for an acute angle.[126]

Unless students have a sufficient number of concepts of a certain order, new concepts of that order cannot be taught directly. One can only give carefully chosen examples of class members from which the student can form his own concept.[127] What has been reported of Bruner's and Dienes' work corresponds with this notion.

Concepts derived from direct sensory experience Skemp calls primary concepts while those that are derived from other concepts are termed secondary. To give someone a new concept it is necessary to arrange for him a group of experiences which have the concept in common, as described in the preceding paragraphs. If it is

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[126]Ibid., p. 7. [127]Ibid.

secondary concept, one has to ensure that the learner grasps those concepts from which the new concept is derived.[128]

In mathematics, "number" and "addition" are primary concepts derived, respectively, from what all collections of a certain number of objects have in common and from what is common to all actions that make two collections into one. The learning of these concepts requires a variety of direct sensory experiences that exemplify them. However, although this approach is suitable for the teaching of primary concepts, it is not suitable for the teaching of secondary concepts. Primary concepts can be exemplified in physical objects, but secondary concepts can only be symbolized. As has already been pointed out, Skemp would agree with Dienes' view that a student must be given a variety of examples from which to form a new concept in his own mind, but, in Skemp's opinion, Dienes' ingenious embodiments of algebraic concepts fail to take into account the difference between primary and secondary concepts. A student may be able to form mathematical concepts, such as those underlying vector algebra, by manipulating concrete embodiments,

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[128]R. R. Skemp, "A Three-part Theory for Learning Mathematics," New Approaches to Mathematics Teaching, F. W. Land, editor (London: Macmillan and Company, 1863), p. 43.

but since, in Skemp's view, mathematical thinking involves reflection upon and formulation of the concepts, such children are only engaged in the first half of the development of mathematical thought structures.[129]

An example of the transition from primary to secondary concepts in mathematics is the generalization of statements like

$$3\times5 + 4\times5 = 7\times5$$

to the statements like

$$3x + 4x = 7x.$$

The understanding of an algebraic statement like the preceding one is derived from discovery of what is common to all arithmetical statements of this form, not what is common to any set of actions with physical objects. Furthermore, an expansion like

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

does not depend on a group of arithmetical statements but on other algebraic concepts symbolized by:

$$(x + a)(x + b) = (x + a)x + (x + a)b$$

$$xb = bx$$

$$ax + bx = (a + b)x$$

$$x.x = x^2$$

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[129] Ibid., pp. 43-44; Skemp, "The Psychology of Learning and Teaching Mathematics," p. 16.

The student thus progresses away from primary concepts as he learns mathematics, and his progress depends on his being able to treat each successive set of secondary concepts as independent of their origins and as generators of the next set. This is possible only if the concepts are well formed and well understood.[130] This description brings to mind and appears to be consonant with Dienes' discussion of recurring learning cycles which reach higher and higher levels of abstraction.

The indirect process of leading a student to form his own concepts may not appeal to many teachers who believe that the only way to ensure that pupils form the right mathematical concepts is to give them clear and unambiguous definitions. However, Skemp has warned that:

> Definitions like "a parallelogram is a plane four-sided figure having its opposite sides parallel" use concepts of the same or higher order than parallelogram, and cannot therefore communicate the concept of parallelogram to a beginner in geometry. The first stage has to be the formation of a set of concepts, by the building-up process already described. Definitions are (among other things) a tidying-up process [*sic*], in which one decides exactly what set of characteristics the concept comprises. That is to say, having formed the concept, it may be possible to *formulate* it--to make the concept the object of consideration in itself, apart from the members of its class from

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[130] Skemp, "A Three-part Theory for Learning Mathematics," pp. 44-45.

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> which it was abstracted [italics not in the original].[131]

A teacher who possesses a certain concept may find it difficult to realize that a pupil does not because, once a concept is formed, it is so obvious to the possessor that it is perceived as part of the data in a situation in which the concept is appropriate. It is difficult to imagine perceiving the data in any other way. An added difficulty arises because pupils can rote-learn all the required manipulations in the early stages so that whether or not they really have developed the required schemata is not readily apparent. However, it is worth the effort to find out since, unless the pupil is provided with the appropriate kinds of repeated experience necessary to form the basic concepts and operations, verbal and blackboard teaching will lead to rote memorization. The concepts required for learning superordinate concepts will not be formed, and the student will never be capable of really understanding mathematics.[132]

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[131] Skemp, "The Psychology of Learning and Teaching Mathematics," p. 8.

[132] *Ibid.*, pp. 9-10; R. R. Skemp, "Reflective Intelligence and Mathematics," *British Journal of Educational Psychology*, 31:50, February, 1960.

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The process of concept formation is the first of the basic concerns on which Skemp's mathematics learning theory focuses. Skemp considers the work of Dienes on concept formation to be particularly insightful, and he recognizes a pressing need for extending Dienes' work up to the secondary school level.[133] There is a similar need for more secondary school work oriented towards Piaget's and Bruner's theories.

In order to progress in learning mathematics one has to build up an organized structure of knowledge of mathematical relationships. Recognizing this necessity, Skemp was led to formulate his notions of "schematic learning."

Schematic Learning. As has been noted previously, contemporary learning theories fail to take into account the way existing knowledge makes possible and influences subsequent learning. In mathematics particularly there is a very strong hierarchical dependence of later learning on previous learning. For example, "solving equations" cannot be learned until all the elementary algebraic processes have been learned and these, in turn, cannot be

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[133]Skemp, "The Psychology of Learning and Teaching Mathematics," p. 10.

learned without knowledge of the arithmetical processes. "Mathematics is probably the most interdependent and hierarchical of any structure of knowledge currently taught."[134]

The Gestaltists have recognized the function of "structures" in learning, but they have located the structure in the material to be learned. Discovery of the "whole" is said to result in an altered perception of all the individual parts of the material, leading to easier learning and better retention. Gestaltists regard discovery of the "whole," of the Gestalt, as an insight into the structure of the material itself. In accordance with Piaget's point of view, Skemp maintains that the true site of the Gestalt is in the learner's mind and that the process of insight is the formation in the learner's mind of a concept (class-concept, relational concept, operational concept, etc.). That the abstracting process occurs in one's mind, rather than pre-existing in perceived material, makes its results available for use with new material and for the solution of new tasks.[135]

> Further, each set of abstractions is available in combination with abstractions derived from

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[134] Ibid., pp. 10-11.

[135] Ibid., p. 11.

> other material encountered at other times, and as a basis for further abstractions. It is this cumulative process especially which Gestalt theory fails adequately to subsume; and which is so central to learning activities which (like mathematics) may extend over periods of the order of ten, twenty, or forty years.[136]

Skemp has noted that Piaget is almost alone among psychologists in insisting on the importance of the existing body of knowledge.[137] As has been mentioned earlier, Piaget refers to the structures of existing knowledge as schemata. The way in which assimilation and accommodation interact to build and preserve the continuity of a schema from its simple beginnings to its final complexity has been discussed on page 47 of this report.

The kind of learning that makes use of and builds more knowledge onto existing schemata Skemp calls <u>schematic learning</u>. Rote learning, in which no structure is made available to give meaning to new material, does not give rise to understanding, to making connections with other learning, to adaptability, to ease of learning, or to retention. Schematic learning does. Students can rote learn to respond correctly, but, when understanding is absent, the schemata required for future learning are not being prepared, with the result that the difficulty of

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[136] <u>Ibid</u>., pp. 11-12.

[137] <u>Ibid</u>., p. 12.

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learning more advanced material eventually becomes insuperable.[138]

Schematic learning can be illustrated by means of the most basic learning schemata: speaking, reading, and writing one's own language.

> Not only is later learning made possible (and also structured) by [the basic schemata], but [the basic schemata] are constantly being practiced and developed during the learning and exercise of more advanced topics.[139]

Skemp sees the concept of schematic learning as fundamental in efforts to advance learning theory to the point at which it can become more directly relevant to classroom learning situations. In addition to its obvious implications for improved teaching methods, consider, for example, that

> . . . the development of certain kinds of schemata can be regarded also as functioning of intelligence. Further development of our understanding along these lines would enable us not only to estimate the extent to which an individual is possessed of intelligence, but also to indicate how this intelligence might most effectively be used for various learning tasks. We could then, perhaps, teach children to use their intelligence intelligently.[140]

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[138] Skemp, "A Three-part Theory for Learning Mathematics," p. 47.

[139] R. R. Skemp, "The Need for a Schematic Learning Theory," British Journal of Educational Psychology, 32:140, 1962.

[140] Ibid., p. 141.

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All of which brings the discussion of Skemp's theory to a consideration of the notion of reflective intelligence, the central construct in the study presently being reported.

Reflective Intelligence. Mathematical thinking is particularly characterized by the process of exploration and generalization that derives new class-concepts and operations from existing ones and that applies class-concepts and operations in fields different from those of their origin. Although ideas do arise from sensori-motor experiences, the development of new mathematical ideas is almost entirely conceptual.[141]

A simple example of mathematical generalization is found in the development of operations with indices. The definitions

$$x^a \cdot x^b = x^{a+b} \text{ and } x^a \div x^b = x^{a-b}$$

are usually arrived at from experiences with such examples as

$$x^2 = x \cdot x, \quad x^3 = x \cdot x \cdot x, \quad x^5 = x \cdot x \cdot x \cdot x \cdot x$$

by the process of concept formation previously described. According to the derivation, the definitions hold only if

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[141]Skemp, "The Psychology of Learning and Teaching Mathematics," p. 13.

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a and b are positive and, for division, if $a < b$, but what if $a > b$? Returning to first principles and using simple algebra,

$$x^3 \div x^5 = \frac{1}{x^2}.$$

But, according to the definition for division using indices,

$$x^3 \div x^5 = x^{-2}.$$

The two operations agree if, by definition, one asserts that

$$\frac{1}{x^2} = x^{-2}.$$

This result can then be further generalized to arrive at the definition

$$x^{-a} = \frac{1}{x^a}.$$

The original definitions for $x^a \cdot x^b$ and $x^a \div x^b$ are derived from examples of such operations, but the extension to include negative indices is made by defining new exemplars to fit the original rules. This latter process is a simple example of reflective activity. Mathematical generalization is a two-part process in which concept formation alternates with reflection on these concepts.[142] It is clear

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[142] *Ibid.*, pp. 13-14, 16.

that Skemp's notion of mathematical generalization, on which his concept of reflective activity is based, is consistent with that of Dienes.

In order to explain the ability to make generalizations, it is necessary to postulate a mental system which is capable of testing whether or not an idea is consistent with one's schemata, of considering a series of mental representations of acts, and of re-arranging and modifying mental representations to arrive at a desired outcome. This system, which enables the mind to turn inward on itself to consider its own concepts and operations, Skemp calls the reflective system. He draws a distinction between this system and the sensori-motor system, the latter of which only enables one to perceive and act on physical objects.[143] A detailed report of Skemp's contrast between reflective activity and sensori-motor activity has been presented on pages 11 to 13 of this report. That Skemp's distinction between sensori-motor activity and reflective activity parallels Piaget's distinction between concrete operations and formal operations is evident.

Of course the reflective system has functions other than the building up of a schema of mathematical concepts.

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[143] Ibid., p. 14.

One of these functions is the application of one's concepts and operations to cope with a particular task, i.e., the solution of problems. A problem is a task which cannot be handled by routine application of known methods but which requires modifications and combinations of existing methods. Successful problem solving requires that the necessary concepts and operations be available and that such operations be amenable to choice and modification in the light of information gained from previous attempts. Intelligent correction of errors in coping with a difficult problem occurs if one uses as information <u>wherein</u> an attempt was unsuccessful in addition to <u>that</u> it was unsuccessful. This involves reflective awareness of one's methods.[144]

Both the formation of mathematical concepts and problem solving activity are cognitive, organizing processes involving awareness of, modification of, and choice from among mentally represented operations. Such reflective activities make possible what is termed logical thought. However, there is another mode of reflective activity, and this mode has an affective function. It acts on wishes and goals and serves to control one's actions in

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[144]*Ibid.*, pp. 16-17.

terms of punishment and reward. Such notions of the affective system are based on Freud's description of the super-ego. Skemp is of the opinion that external reward and punishment of sensori-motor activity organize one's behaviour patterns into a habit-strength hierarchy in which only the response having the greatest habit strength is available at any one time. Internal reward and punishment produce a similar hierarchical organization, which contrasts sharply with the flexible, multi-response organization built from cognitive modes of reflective thought. The cognitive system examines mistakes and corrects them to build more adequate concepts and processes, but mistakes viewed affectively are considered moral faults and are repressed rather than being corrected. A child dominated by the affective system will show only his successful work to the teacher, whereas the cognitively oriented child will show all of his work, especially his mistakes. One is seeking approval while the other is seeking information. "Wrong" can be interpreted as a value judgement (affective meaning) or as "not consistent with existing information; not leading to an expected result" (cognitive meaning). Which meaning is taken affects one's thinking a great deal. Teachers should be careful to convey which meaning of wrong is intended. A mistake in mathematics is not a moral fault,

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and one's tone of voice should not indicate to the student that it is, lest the wrong kind of reflective activity be encouraged. It is thought provoking to realize that a student who is poor in mathematics and who reacts affectively to it is almost certain to face punishment whether he completes his lessons or not.[145]

If the necessity of dissociating one's thought from affective considerations and Piaget's view that objective thinking can only occur if one superimposes other points of view on one's own are added to the preceding discussions, one comes close to a complete description of Skemp's notions of the development of reflective intelligence, which he has encapsulated as follows:

> Intelligence thus involves successive stages of dissociation of . . . ideas: first from the sensory and motor activity on which they were founded; then, from particular wishes and needs; next, from the particular viewpoint of the individual; and finally, even from the context of the ideas, leaving only the abstractions, forms, and relationships.[146]

While it is clear that Skemp's theory is largely built upon Piaget's theory, there is an aspect of Skemp's notions of reflective intelligence that perhaps extends those of Piaget. According to Piaget:

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[145]Skemp, "An Investigation into Difficulties in Learning Mathematics by Children of Good General Intelligence," pp. 110-125.

[146]*Ibid.*, p. 125.

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> There are . . . three essential conditions for the transition from the sensori-motor level to the reflective level. Firstly, an increase in speed allowing the knowledge of the successive phases of an action to be moulded into one simultaneous whole. Next, an awareness, not simply of the desired results of action, but its actual mechanisms, thus enabling the search for the solution to be combined with a consciousness of its nature. Finally, an increase in distances, enabling actions affecting real entities to be extended by symbolic actions affecting symbolic representations and thus going beyond the limits of near space and time.[147]

For a detailed account of Skemp's characterization of sensori-motor intelligence and reflective intelligence, the reader is referred to pages 11 to 13 of this report. A brief statement of that characterization follows. Sensori-motor intelligence is defined as the perception of relationships between objects and groups of objects presented to the senses and between one's own actions with objects. On the other hand, Skemp defines reflective intelligence as the functioning of a second order mental system which can perceive relationships among and act upon the concepts and operations of the sensori-motor system, taking into account their relationships as well as information from the memory and the external environment. Piaget's description refers to the reflective system's awareness of the mental representations of the sensori-motor system but not explicitly

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[147]Jean Piaget, The Psychology of Intelligence (London: Routledge and Kegan Paul, 1950), p. 121.

to the reflective system's ability to reorganize, to select, and to alter these mental representations. Reflective awareness alone is not sufficient to account for deliberate modifications of, and choices from among, the mentally stored concepts and operations.[148]

However, in the present context it is more important that Skemp has interpreted Piaget's ideas in such a manner that they seem directly relevant to and clearly applicable to the teaching of secondary school mathematics. The bulk of Piaget's work, as well as that of Bruner and Dienes, has been concerned with the thinking of elementary school children. Much more needs to be done at the secondary school level.

It is quite apparent that practically every assertion Skemp makes on the basis of his theory, with the possible exception of his explanation of the way anxiety affects one's thinking processes, can be supported by reference to the theories of Piaget, Bruner, and Dienes. One interpretation of Skemp's theory, which emerges from his published articles and which has guided the discussion in Chapter I of this report, emphasizes the contrast

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[148] Skemp, "Reflective Intelligence and Mathematics," pp. 48-49; Skemp, "An Investigation into Difficulties in Learning Mathematics by Children of Good General Intelligence," pp. 102-103.

between sensori-motor intelligence and reflective intelligence. Piaget has used these terms in his own writings,[149] but his notions of sensory-motor, pre-operational, concrete operational, and formal operational levels of cognitive development are more widely discussed. It is clear that what Skemp refers to as sensori-motor intelligence would be characteristic of the thought processes of the concrete operational child. One might say that the "sensori-motor" label for this type of thinking, while natural, nevertheless conflicts with Piaget's more widely known use of the term in connection with the thinking patterns of an infant. However, this is a minor point which need not present difficulties. It is apparent that Skemp's conceptions of reflective activity and reflective intelligence closely correspond with Piaget's notions of formal thought. Skemp's assertion that he has added something to Piaget's description in his discussion of reflective intelligence seems somewhat more illusory than real in the light of an examination of Piaget's whole theory. That one can choose, manipulate, and modify the processes imbedded in one's schemata does seem to be treated in general discussions of Piaget's theory. A point that might be raised is that

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[149]Piaget, The Psychology of Intelligence.

Skemp appears to invest intelligence with some special faculty, which Piaget does not. However, this impression may result merely from Skemp's choice of terms in describing his notions.

Another interpretation of Skemp's theory, which arises from a detailed examination of the development of his ideas as he describes it in his doctoral dissertation,[150] places less emphasis on the distinction between sensori-motor and reflective intelligence. Rather, one finds the stages in arithmetical thinking to be described in a manner that parallels the description of the stages in mathematical thinking in general. This interpretation, based as it is on notions of the formation of new concepts and processes and the formulation of these to prepare the way for formation of higher order concepts and processes, seems amenable to Bruner's and Dienes' notions about learning proceeding in recurring cycles that are microcosms of the overall developmental sequence. That is to say, depending on whose terms one uses, in learning one proceeds through cycles of enactive, iconic, and symbolic stages, or through cycles of constructive and analytic activity. The

[150] Skemp, "An Investigation into Difficulties in Learning Mathematics by Children of Good General Intelligence."

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second interpretation of Skemp's theory does lend itself to such cyclic notions because the development of mathematical ideas is interpreted in the same way as development of arithmetic ideas, only at a higher level of abstraction and generality. It was the published interpretation of Skemp's theory that led Dienes to remark:

> As regards mathematics learning, Skemp's hypothesis is that in the learning of arithmetic mostly the sensory motor system is involved whereas in the learning of algebra and the higher reaches of mathematics the reflective system is involved. One cannot help thinking that this hypothesis would not have been made, had arithmetic not been treated as a mechanical set of stimulus and response situations, which simply had to be learned.[151]

This comment is certainly less relevant to the interpretation of Skemp's theory as originally discussed in his dissertation because he does describe arithmetic problem solving and even higher order concept formation at the arithmetic level in terms of the functioning of reflective intelligence. However, one cannot avoid the observation that Dienes' notions of constructive thinking processes are less restrictive than those of Skemp regarding the functioning of sensori-motor intelligence. Constructive thinking can be engaged in at any level of abstraction, but, according to Skemp's formulation, sensori-motor

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[151] Z. P. Dienes, Mathematics in Primary Education (Hamburg: International Study Group for Mathematics Learning, 1966), p. 48.

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intelligence can act only on concrete reality. Here the notions of Piaget and Skemp are very close. However, as has been discussed previously, it is not difficult to reinterpret Piaget's developmental theory, and hence Skemp's, in terms of the cycles of Bruner and Dienes. In any event, Skemp's notions of reflective thinking closely parallel those of Bruner regarding symbolic thinking and those of Dienes regarding analytic thinking. The higher levels of constructive thought in Dienes' system can be discussed in terms of aspects of reflective thinking in Skemp's system.

Skemp's description of schematic learning is distinctly Piagetian as well as being assimilable to Bruner's "coding systems." The application of schematic learning notions to mathematical learning situations seems potentially fruitful as an operational construct.

Skemp's notions of concept formation agree very closely with Bruner's insights in this area. The direct application of these notions to mathematics learning situations by Skemp produces clear insights regarding the relevance of such notions for classroom teachers of mathematics. In fact, the outstanding feature of Skemp's entire theory is its common sense relevance and applicability to secondary school mathematics learning and teaching. Its empirical relevance requires further substantiation.

Now that Skemp's theory has been described in terms of the ways it is different from and similar to other attempts to describe the nature of learning, the focus of the review turns from the theoretical background of the study presently being reported to a consideration of relevant empirical research that has been completed.

IV. REVIEW OF RESEARCH ON SECONDARY SCHOOL MATHEMATICS LEARNING

Research having a direct bearing on the present study appears to be relatively scarce, partly because of the rather specific focus of the study in terms of Skemp's theory and partly because investigations into the nature of mathematics learning at the secondary school level have been infrequent. Nevertheless, a number of studies considered relevant in the present context are briefly reviewed in succeeding paragraphs under five headings: Piagetian Studies in Secondary School Settings, Skemp's Reflective Intelligence and Schematic Learning Studies, Skemp's Anxiety Studies and Related Research, Prediction Studies, and Factor Analysis Studies. Perhaps noticeable by their absence are reviews of Bruner's and Dienes' research. Bruner's recent cognitive growth studies are somewhat more psychologically oriented and less classroom oriented than

is considered appropriate for discussion in the present context, and they generally involve children of elementary school age. Dienes' studies, though definitely oriented towards mathematics education, are almost exclusively carried out in elementary school situations, and descriptions of the studies are very closely tied to Dienes' discussions of the development of his theory. Consequently, it was decided that further discussion of Bruner's and Dienes' research at this point would add very little to the discussions of their work presented earlier in this chapter. However, the Piagetian studies reviewed were included because they involved secondary school students and because the discussions engendered by them suggest interpretations and applications of Piaget's theory that complement and extend previous discussions of the theory. The prediction and correlation studies reviewed, while representative of work being pursued in this area, are presented mainly to support decisions made regarding the battery of tests and the statistical procedures used in the present study.

Piagetian Studies in Secondary School Settings

Since a central concern of the present report is to characterize the thinking of secondary school students in mathematical situations, the experiments of Piaget and

collaborators that seem most relevant are those designed to illustrate contrasts between formal operational thought and concrete operational thought. The sixteen experiments described in The Growth of Logical Thinking from Childhood to Adolescence[152] were explicitly designed for this purpose. Since in the present context the details of the clinical method of the Geneva group is of secondary interest, the present review is restricted to somewhat terse descriptions of selected experiments and their general findings.

In the first experiment described in The Growth of Logical Thinking, the subjects were given the task of shooting a ball out of a plunger that could be aimed so that the ball would hit a target on the table after rebounding from a cushioned bank. Each subject was questioned about what he observed to see if he was able to induce from his various attempts that the angle of incidence equals the angle of reflection. Concrete-operational subjects were found to be limited to asserting concrete instances of the relation and to making practical use of it to shoot accurately; they were unable to state it in its general form as a law. On the other hand, adolescents seemed to

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[152] Bärbel Inhelder and Jean Piaget, The Growth of Logical Thinking from Childhood to Adolescence (New York: Basic Books, Inc., 1958).

look for general principles from the outset, forming general hypotheses about the regularities they observed and putting them to experimental test.[153]

Another experiment required each subject to explain why bodies of various densities and sizes would float or sink in water. Concrete-operational children would try to arrive at an explanation by means of a double-entry classification of observations: large-heavy, small-heavy, large-light, and small-light. They would identify the class of small-heavy objects as the non-floating class. They did not arrive at a concept of density because they could not think in terms of the volume of water displaced to relate the weight of the object to the weight of water displaced for the only volumes that could be empirically observed were the volume of the object and the complete volume of water in the container--there is no empirical correlate to density for the concrete operational child. However, formal-operational subjects, about age eleven or twelve, were able to eliminate contradictions by casting their explanations in terms of an integrated system of variables. Rather than a double-entry classification, the

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[153]Ibid., pp. 3-19; John H. Flavell, The Developmental Psychology of Jean Piaget (New York: D. Van Nostrand Company, Inc., 1963), pp. 347-348.

formal-operational subject would utilize a logical structure involving reciprocal implication and the notion of independence between two variables. He eventually would postulate that a given object floats only if its weight is lighter than that of an equal volume of water.[154]

The adolescent's growing skill in scientific reasoning was illustrated by an experiment in which the problem was to discover the variables affecting how much a rod would bend under a given set of conditions. The materials involved and procedures employed were such that it was possible to isolate five variables as affecting the amount of bending of any particular rod: the type of metal of which the rod was made, the amount of weight supported, the length of rod, the rod's thickness, and its cross-sectional form (round, square, or rectangular). Most adolescents succeeded in differentiating the five variables. Using their combinatorial ability, they systematically tested most or all variable-present, variable-absent combinations. For example, they might vary thickness, holding the other variables constant. Younger subjects could discover some of the variables, and they did make crude attempts to test their effects, but they were unable to employ the "all-

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[154]Inhelder and Piaget, _op. cit._, pp. 20-45; Jerome S. Bruner, "Inhelder and Piaget's _The Growth of Logical Thinking_: I, A Psychologist's Viewpoint," _British Journal of Psychology_, 50:366-367, 1959.

other-things-being-equal" method to demonstrate the individual effect of each variable. The disposition for systematic proof seems to be the special domain of the formal-operational thought structure.[155]

The experiment involving colourless liquids, some combination of which would produce a colour, was referred to on page 57 of the present report. In this experiment, it was found that concrete-operational children did test "1 by n" and "n by n" combinations (logical multiplication), but they did not do so systematically to find out which combination(s) would produce the colour. They could not systematically eliminate variables. In contrast, formal-operational subjects generated systematic combinatorial tests to eliminate those combinations that were not adequate.[156]

The experiments just described and the others in _The Growth of Logical Thinking_ served to demonstrate that the significant difference between adolescent and pre-adolescent thinking is the ability to operate propositionally rather than concretely. The adolescent's formal

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[155]Inhelder and Piaget, _op. cit._, pp. 46-66; Flavell, _op. cit._, p. 348.

[156]Inhelder and Piaget, _op. cit._, pp. 107-122; Bruner, "Inhelder and Piaget's _The Growth of Logical Thinking_," 367-368.

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thinking structures enable him to get past the errors inherent in limited concrete tests by employing the sixteen binary operations of formal logic (affirmation, negation, conjunction, disjunction, implication, etc.) to generate systematic tests by which relevant variables can be isolated.[157]

K. Lovell[158] has repeated ten of the sixteen experiments described in The Growth of Logical Thinking. Each of two hundred subjects, who were mostly between the ages of eight and eighteen, was examined individually on some selection of four of the ten experiments, with everyone doing the combinations of colourless liquids experiment described previously. Piaget's clinical approach was used and, on the basis of the protocols so collected, the performance of each student was ranked according to nine stages: one stage of pre-operational thinking, four stages of concrete thinking, and four stages of formal thinking. The results confirmed the existence of the three main stages in the growth of logical thinking as proposed by Inhelder and Piaget. Support was found for the contention that only rarely do pre-adolescents reach the stage of

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157 Ibid., p. 368.

158 K. Lovell, "A Follow-up Study of Inhelder and Piaget's The Growth of Logical Thinking," British Journal of Psychology, 52:143-153, 1961.

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formal thinking and that the ablest adolescents, though not all adolescents, do exhibit formal thinking processes. Some of the evidence suggested that the least able adolescents do not pass beyond the concrete operations stage.[159]

Analysis of the rankings for each student from the four experiments participated in showed considerable agreement among the levels of thinking displayed in each of the experiments. Where the kinds of ideas encountered in some of the experiments overlapped school experiences only a minimal effect on the rankings was observed. Examination of the protocols led the investigators to conclude that instruction seemed to have been of greatest value when the required thinking skills were almost or actually available to the subject.

> If the power to think at the requisite level is not present, knowledge gained by instruction is either forgotten, or it may remain rote knowledge and be regurgitated when required.[160]

The experiments were found effectively to separate students classified as fast or slow learners in a wide variety of school subjects, leading the experimenters to conclude that the types of thinking processes involved in the

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[159]Ibid., pp. 143-149. [160]Ibid., p. 151.

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Piagetian experiments are broadly applicable rather than only being relevant to scientific or technical problems.[161]

To set the stage for another Piagetian replication study, consider that Piaget has found that seventy-five per cent of the subjects he has tested have evidenced conservation of substance by age seven to eight, conservation of weight by age nine to ten, and conservation of volume by age eleven to twelve. Since the latter finding related to children in the age range of the concern of the present report and since David Elkind's first replication of the relevant Piagetian experiments showed that only twenty-seven per cent of a group of eleven to twelve-year-old American children had attained conservation of volume,[162] a brief account of the findings of Elkind's replication study with twelve to eighteen-year-olds is reported in succeeding paragraphs.

Elkind's second replication study was designed to determine the influence of age, sex, and IQ on the attainment of abstract conceptions of quantity (conservation

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[161] Ibid., pp. 149-153.

[162] David Elkind, "Quantity Conceptions in Junior and Senior High School Students," Child Development, 32:551, 1961.

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of quantity; judgement of sameness despite perceptual change) in adolescents in addition to extending the replication of Piaget's experiment to the twelve to eighteen-year-old age group. Four hundred sixty-nine Massachusetts junior and senior high school students with a mean IQ of 100.4 were given group tests of conservation (Piaget's experiment and Elkind's first replication had used individual tests.). The materials used were two identical one and one-half inch clay balls and a small balance scale. Tests for conservation of mass, weight, and volume were given in that order. In the test for conservation of mass, it was explained that the two balls were identical in every way and that there were no tricks in the experiment. Several students were asked to verify that the two balls weighed the same by using the scale. Any students with doubts were asked to voice them, and the use of "same amount" was clarified so that all students agreed that the balls were the same. The students were then asked:

> (a) "Do the balls both contain the same amount of clay?" (identity question); (b) "Suppose I make one of the balls into a sausage, would the two pieces of clay still contain the same amount of clay?" (prediction question); . . . [after one of the balls had actually been rolled into a sausage] (c) "Do they both contain the same amount of clay now?"

(judgment question); (d) "Explain your answers." (explanation question).[163]

The test for conservation of weight repeated the above procedure with "weight" substituted for "amount" (after the experimenter had rolled the sausage back into a ball shape). The test for conservation of volume followed the same procedure except that "volume" and "same room or space" were used instead of "amount." A test was considered passed only if the subject answered all four questions (identity, prediction, judgment, and explanation) so that conservation was clearly evident. As a special check on the conservation of volume test, students were asked what would occur if the ball and sausage were placed in identical glasses filled equally high with water.[164]

As verbal misunderstandings were carefully avoided by using the experimental techniques described above, it was asserted that students who failed any of the conservation tests did so solely because of inadequate conceptions. Those who failed the volume test had agreed that the balls were initially the same, but they predicted, judged, and explained that changes in shape produced changes in volume. [One cannot help wondering if this

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[163] Ibid., p. 552

[164] Ibid., pp. 550-553.

might not result from the common experience of squashing a container only to find that it no longer holds as much as it formerly did. Could there be confusion between the internal capacity of a container and the volume of substance in a solid?] Most of these same students had previously asserted that mass and weight were conserved because nothing was added or taken away, because changing the shape did not change the amount, or because what the sausage lost in width it gained in length. They failed to generalize their notions regarding mass and weight to the situation involving volume, which they treated as an entirely different problem--they failed to dissociate their subjective, sensori-motor conceptions from their objective, logico-mathematical conceptions.[165]

Of the students tested, eighty-seven per cent demonstrated conservation of mass and weight, but only forty-seven per cent had abstract conceptions of volume. In fact, only in the oldest age group (mean age: 17.7 years) did more than seventy-five per cent exhibit conservation of volume. Piaget's finding that eleven to twelve-year-olds had attained conservation of volume was again not confirmed. However, a steady increase with age in the

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[165] Ibid., pp. 553-554.

percentage of students exhibiting conservation of volume was observed (thirty-eight to seventy-nine per cent for boys in groups with mean ages of 12.6 to 17.7; twenty-six to sixty-eight per cent for girls in groups with mean ages of 12.6 to 17.7). At each age-level a significantly higher percentage of boys than that of girls exhibited conservation of volume (The observed chi-square was greater than that required for significance at the 0.01 level.). A low, but positive, point biserial correlation (0.31) between IQ and success in the volume test was calculated.[166]

According to Piaget's theory, the majority of children of eleven or twelve should be _ready_ to attain conservation of volume because this conceptualization requires only concrete operations, which are present in most children by age seven, and because they have had sufficient concrete experience to form abstract conceptions of mass and weight, the structural prerequisites for conservation of volume. However, the age at which the child (adolescent) is _ready_ to grasp conservation of volume is also the age at which formal operations are developing. Whereas concrete operations are concerned with immediate reality, formal operations are concerned with construction of systems and

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[166] _Ibid._, pp. 554-555.

theories designed to investigate possibility. According to Elkind, the appearance of formal operations thus produces new interests which tend to reduce one's concern with inductive conceptualization from the physical environment in favour of more theoretical interests. The possibility of spontaneous discovery of the conservation of volume is thus reduced. The adoption of adult roles, beginning about age eleven or twelve, also leads the adolescent to be more selective in his choice of experiences. For example, the prospective scientist would likely choose different experiences from those chosen by the aspiring mechanic. It would seem reasonable to conjecture that those adolescents who attain conservation of volume despite decreased motivation have simply adopted roles conducive to the formation of quantity conceptions. On the other hand, though ready, many would not attain conservation of volume because their roles do not provide the necessary experience.[167]

Similarly, the increase with age in the percentage of students attaining conservation of volume can be partly accounted for by realizing that those who remain in school are more likely to have adopted professional or academic occupational roles, which would differ from nonprofessional roles in providing experiences that would lead students to

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[167] Ibid., pp. 556-557.

the abstract conceptualization of volume. The greater percentage of boys attaining conservation of volume is also consistent with the hypothesis about the effects of various roles. Considering that the mean IQ of the girls in the experiment was somewhat higher than that of the boys and that there was no significant difference between the sex groups with respect to attainment of mass and weight conservation, the difference with respect to attainment of volume conservation cannot be attributed to innate differences in conceptual ability between the sexes. It seems reasonable to conjecture that the boy's traditional role of social ineptness but scientific expertise as contrasted with the girl's role of social skill and scientific aversion would give more boys than girls the opportunity to gain an abstract conception of volume.[168]

Skemp's Reflective Intelligence and Schematic Learning Studies

One of Skemp's experiments was designed to test the hypothesis that the logical multiplication of concepts is an important ability related to success in mathematics. The method used was adapted from a card sorting experiment

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[168]*Ibid*., p. 558.

described by E. A. Berg[169] (Bruner's _A Study of Thinking_ was not available in England when Skemp's experiment was being conducted.). A pack of sixty-four cards was prepared, each card having one, two, three, or four figures (crosses, circles, squares, or triangles) coloured red, yellow, blue, or green. Four cards depicting, respectively, one red triangle, two green squares, three yellow crosses, and four blue circles were laid in a row in front of the subject, one of twenty-five third form (thirteen or fourteen-year-old) grammar school boys involved in the experiment. Each subject was given three simple classification problems in which he was given the remaining sixty cards and asked to place them one at a time in front of one of the four cards on the table to try to find out what sorting criterion the experimenter had in mind (classification by colour, number, and form, respectively, for the three simple problems). Each time a subject laid down a card his response was recorded and he was told whether his classification was "right" or "wrong." When a subject had sorted five consecutive cards correctly he was told that he had solved the problem. The experimental group (13 boys) was asked

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[169]Esta A. Berg, "A Simple Objective Technique for Measuring Flexibility in Thinking," _Journal of General Psychology_, 39:15-22, 1948.

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to describe the method of sorting used, but the control group (the other 12 boys) was not asked to do this. After the third problem, the experimental group was asked to explain <u>how</u> they had discovered the correct sorting methods in the three problems. Then a blank card was added to the row on the table, the subject's attention was drawn to it, and he was told that the sorting criterion in the fourth problem would be a combination of two of the criteria used in the first three problems. Any of the cards that could not be matched to cards on the table in terms of the double criterion (colour and number, number and form, or form and colour) were to be placed under the blank card. The choice of the double category was determined by the subject's first match that agreed with any of the three possible double categories. For this problem the subject's sorting was allowed to continue until it was clear that he had grasped the double category (For cards placed under the blank card it was not clear whether the problem had been solved.). The subject's score was taken as the number of trials required before the subject could correctly sort five or more consecutive cards.[170]

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[170]R. R. Skemp, "An Investigation into Difficulties in Learning Mathematics by Children of Good General Intelligence" (unpublished Doctoral Dissertation, The University of Manchester, 1958), pp. 144-148.

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The mean score of the experimental group was 14.3 trials while the control group required a mean of 21.8 trials to solve the problem. However, the difference was not statistically significant. An interesting further result was obtained by dividing the experimental group into two categories of seven and six subjects, respectively, according to whether or not the explanation of <u>how</u> the correct sorting techniques were discovered for the first three problems showed clear awareness of the solution method used. The mean score of the method-aware group was 2.4 trials as compared with a mean score of 28.0 trials for the others, a statistically significant difference. The latter finding tends to support the contention that clear awareness of one's methods helps one to build on them, whereas the former finding indicates that just verbalizing one's method may be only one factor in producing explicit consciousness of it.[171]

Using the scores of both the experimental and control groups, a product-moment correlation of 0.52 between the subjects' scores on the compound problem and their school mathematics marks was calculated. The correlation was significantly different from zero (0.05 level test), which

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[171] *Ibid.*, pp. 148-151.

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lends support to the hypothesis that ability to combine concepts and to transfer methods from one problem to another (i.e., the ability to use reflective processes) is related to ability in mathematics.[172]

Skemp considered the foregoing to be a pilot experiment which had the effect of supporting his belief that reflective processes are important in mathematics. Consequently, he designed an experiment to measure student ability to form and to manipulate concepts and operations. He emphasized manipulation in the design of the study because conscious manipulation of concepts and operations requires that these be formulated, a reflective process, whereas the formation of concepts and operations may or may not require reflective activity. Furthermore, he considered that the chief ability required in mathematics at the secondary school level is the skillful combination and use of known concepts and operations. Accordingly, Skemp developed two two-part tests to measure student ability to form and reflectively manipulate concepts and operations.[173]

For the test involving concepts, fifteen properties were chosen that could be possessed or not by simple line

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[172] Ibid., p. 151.

[173] Ibid., pp. 156-157.

drawings (e.g., being curved, continuous, closed, dotted, or self-crossing). For the first part of the test (labelled SK4, Part I), Skemp drew three exemplars (labelled "Examples"), three non-exemplars (labelled "Not Examples"), and three "Test Figures" for each of the fifteen properties. The subjects were asked to try to discover the property held in common by the "Examples" and to indicate which of the "Test Figures" possessed this property. The criterion for SK4, Part I was thus the ability of the subject to use the concept (indicating he had formed the particular class-concept) rather than his ability to verbalize or formulate it. Since the second part of the test was to be a measure of the student's ability to manipulate the concepts of the first part, a preliminary trial with twelve-year-olds was carried out, and only those concepts which most children were able to grasp were retained.[174]

The second part of the concepts test (SK4, Part II) was designed to measure the student's ability to manipulate concepts. For this part, thirty-five pairs of the properties used in the first part were chosen, and three exemplars of each double property were drawn. The three non-examplars drawn for each double property had, respectively, only one of the properties, the other property,

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[174] Ibid., p. 157.

[illegible]

and neither property. The subjects were asked to decide whether or not each of three "Test Figures" possessed both properties. To grasp and demonstrate each double concept the subject would have to think reflectively, not only being aware of the single class-concepts but deliberately combining and separating them.[175] (A reproduction of SK4, Parts I and II, is included in the Appendix.)

The operations formation part of the second test (SK5 A) consisted of an answer sheet and a demonstration sheet giving three simple abstract-line-figure examples of each of fifteen operations (e.g., clockwise rotation through a right angle, reflection in a horizontal line, interchanging the numbers of figures in two groups). The subjects were asked to discover what each operation was from the demonstration sheet (after some similar operations had been explained) and then to carry out the operation on three specified figures on the answer sheet, drawing the results in provided blank spaces.[176]

The second part of the operations test, manipulation of operations (SK5 B), involved combining and reversing the operations from SK5 A. Both of these processes involve reflective activity and are relevant to mathematics, in

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[175] Ibid., p. 158. [176] Ibid.

which practically every operation has its reverse (or inverse) and in which successful problem solving depends on suitable choices from among and combinations of one's available operations. Of the fifteen problems given, the first five involved carrying out two operations in combination (one after the other) on three figures and showing the results as in SK5 A. The second group of five problems involved carrying out the reverse of a single operation on three figures, and the third group of five involved conceiving of the reverses of two operations and then carrying the reverses out in combination (one after the other). Before beginning SK5 B, the subjects were given the answers to SK5 A (after their own answers had been collected) and these were explained to ensure that the basic operations were understood. The demonstration sheets were used by the students in SK5 B, as in SK5 A, so that no memorizing was required.[177] (A reproduction of a modified version of SK5 A and B, viz., SK6, Parts I and II, is included in the Appendix.)

The scores on the formation parts of the two tests were crowded near the maxima, as intended. Not all of the subjects who wrote the SK4 test were available for the SK5

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[177]Ibid., p. 159.

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test. Consequently, 276 third, fourth, and fifth form students (twelve to sixteen-year-olds) wrote the former whereas 230 wrote the latter. For the third and fourth forms the best available mathematics criterion was the division of the classes into streams according to mathematics achievement. Table I, on page 152, summarizes the results according to mathematics stream and shows that both measures of reflective ability appear to discriminate between streams. The manipulation of operations test, SK5 B, shows more marked discrimination than the manipulation of concepts test, SK4, Part I. Both the discriminating power and the differences in this power between the two tests tend to support Skemp's hypothesis that reflective activity, especially that with operations, is important in mathematics.[178]

Fifty of the fifth form students had completed both parts of SK4 and SK5, and they had all written the same general certificate of education (G.C.E.) mathematics exams. The correlations among the G.C.E. mathematics scores and the scores on Skemp's tests are shown in Table II on page 153.[179] The highest correlation is that between the manipulation of operations (SK5 B) scores

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[178] _Ibid._, pp. 160-161. [179] _Ibid._, p. 162.

TABLE I

MEAN MANIPULATION OF CONCEPTS (SK4, PART 2) AND MANIPULATION OF OPERATIONS (SK5 B) SCORES OBTAINED BY THIRD, FOURTH, AND FIFTH FORM STUDENTS IN VARIOUS MATHEMATICS STREAMS*

Form	Math Stream	N	Mean SK4, Part 2 Score	Mean SK5 B Score	N	Math Stream	Form
III	1	31	23.6	33.7	26	1	III
	2	32	19.4	25.2	29	2	
	3	30	16.7	21.6	30	3	
IV	1	26	25.9	32.2	22	1	IV
	2	24	20.1	29.9	23	2	
	3	24	23.5	24.5	22	3	
	4	21	18.2	20.8	20	4	
V	1	23	26.7	38.2	19	1	V
	2	21	24.2	34.3	12	2	
	3	22	19.7	29.2	18	3	
	4	22	17.1	21.8	9	4	
TOTAL N		276			230		

*R. R. Skemp, "An Investigation into Difficulties in Learning Mathematics by Children of Good General Intelligence" (unpublished Doctoral Dissertation, The University of Manchester, 1958), p. 161.

TABLE II

CORRELATIONS AMONG SCORES OBTAINED BY FIFTY FIFTH FORM STUDENTS ON SKEMP'S TESTS AND ON A G.C.E. MATHEMATICS EXAM*

	SK4, Part 2	SK5 A	SK5 B	G.C.E. Mathematics
SK4, Part 2	1.00	0.66	0.68	0.58
SK5 A	0.66	1.00	0.58	0.42
SK5 B	0.68	0.58	1.00	0.72
G.C.E. Mathematics	0.58	0.42	0.72	1.00

*R. R. Skemp, "An Investigation into Difficulties in Learning Mathematics by Children of Good General Intelligence" (unpublished Doctoral Dissertation, The University of Manchester, 1958), p. 162.

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[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

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and the mathematics scores, supporting Skemp's hypothesis that reflective activity is important in mathematics. The fairly high intercorrelations between SK4, Part II, SK5 A, and SK5 B, as well as between these and the mathematics criterion could perhaps be attributable to ability to deal with abstract material of the kind found in Skemp's tests. It might be argued then that the high correlations of scores on Skemp's tests with mathematics scores demonstrates that mathematics involves the ability to deal with abstract material rather than the ability to use reflective processes. Since each of Skemp's tests do involve similar kinds of abstract material but different processes (non-reflective and reflective), partialing out of that part of the correlation between Skemp's tests and mathematics that is attributable to such commonality should remove any of the correlation due only to the abstract nature of the material. Table III, on page 155, displays the partial correlations that Skemp calculated.[180]

As can be seen from Table III, the effects of partialing out either or both of the SK4, Part II, and SK5 A portions of the correlation between SK5 B and the G.C.E. mathematics criterion does not substantially reduce the

[180]*Ibid.*, pp. 162-163.

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TABLE III

PARTIAL CORRELATIONS AMONG SCORES OBTAINED BY FIFTY FIFTH FORM STUDENTS ON SKEMP'S TESTS AND ON A G.C.E. MATHEMATICS EXAM*

Code:
1 -- SK4, Part 2, Manipulation of Concepts
2 -- SK5 A, Operations Formation
3 -- SK5 B, Manipulation of Operations
4 -- G.C.E. Mathematics Exam

$r_{34.1}$** = 0.61

$r_{34.2}$ = 0.64

$r_{34.12}$ = 0.55

$r_{24.1}$ = 0.06

$r_{24.3}$ = 0.01

*R. R. Skemp, "An Investigation into Difficulties in Learning Mathematics by Children of Good General Intelligence" (unpublished Doctoral Dissertation, The University of Manchester, 1958), p. 163.

**The notation $r_{34.1}$ stands for the partial correlation between variables 3 and 4 after the effects of variable 1 have been removed.

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[illegible]	[illegible]	[illegible]

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original correlation of 0.72. However, the correlation between the mathematics criterion and SK5 A, which involves dealing with abstract relationships and, also, to an unknown extent, reflective processes, is virtually reduced to zero when either of the portions attributable to relationships with the tests definitely dependent on reflective processes is partialed out. From these results Skemp concluded that the intercorrelations shown in Table I can be ascribed to varying degrees of dependence of the scores on the functioning of reflective processes rather than on the abstract nature of the material.[181]

When the fourth form students had progressed to the fifth form and had written a set of G.C.E. examinations a year after writing Skemp's tests, Skemp was able to calculate correlations between the students' scores on his tests and their mathematics scores. For eighty-eight of the students originally in the fourth form, the correlations between the G.C.E. mathematics scores and scores on the SK4, Part II, SK5 A, and SK5 B tests were, respectively, 0.56, 0.48, and 0.73. These figures show remarkable agreement with the correlations calculated for the students originally in the fifth form. Considering the

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[181] _Ibid._, p. 164.

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nature of Skemp's tests, the predictive power of SK5 B, especially, in terms of success in mathematics, appears to be exceptionally good. Skemp has also calculated odd-even item correlations to arrive at reliability coefficients of 0.76, 0.94, and 0.95 for SK4, Part II, SK5 A, and SK5 B, respectively, by pooling the responses from the fourth and fifth form students (N = 138).[182]

For those students originally in the fifth form, Skemp calculated some multiple correlations between his test scores and the mathematics criterion. The multiple correlation between the mathematics criterion and an optimum weighting of SK4, Part II, and SK5 B scores was 0.77 for the total group, 0.90 for the girls only, and 0.88 for the boys only.[183]

Another experiment conducted by Skemp[184] was designed to demonstrate the crucial importance of schematic

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[182] R. R. Skemp, "Reflective Intelligence and Mathematics," British Journal of Educational Psychology, 31:53, February, 1960.

[183] Skemp, "An Investigation into Difficulties in Learning Mathematics by Children of Good General Intelligence," p. 223.

[184] R. R. Skemp, "The Need for a Schematic Learning Theory," British Journal of Educational Psychology, 32:133-142, 1962.

learning even for a straightforward learning task less difficult than those usually required of pupils in school. Two different artificial schema, labelled Schema I and Schema II, were devised, each based on sixteen symbols and their associated meanings. (A sampling of symbols from Schema I is presented in Figure 1 on page 159.) These symbols were combined in pairs, groups of three, and successively larger groups to symbolize new meanings based on the meanings of the basic symbols and groups of basic symbols.[185]

Two parallel groups of first form (eleven to twelve-year-old) students, twenty-three in one group and twenty-four in the other, participated in the experiment. Group I learned Schema I by memorizing basic word-symbol pairs and successively larger symbol groups with their corresponding word meanings. Group II learned Schema II in the same way. A set of multi-symbol groups consisting of from eight to fourteen symbols was prepared for each of the schema. Both groups were asked to learn material assimilable to Schema I (schematic learning for Group I; rote learning for Group II) by learning to associate each group of symbols with a particular word, and then to learn

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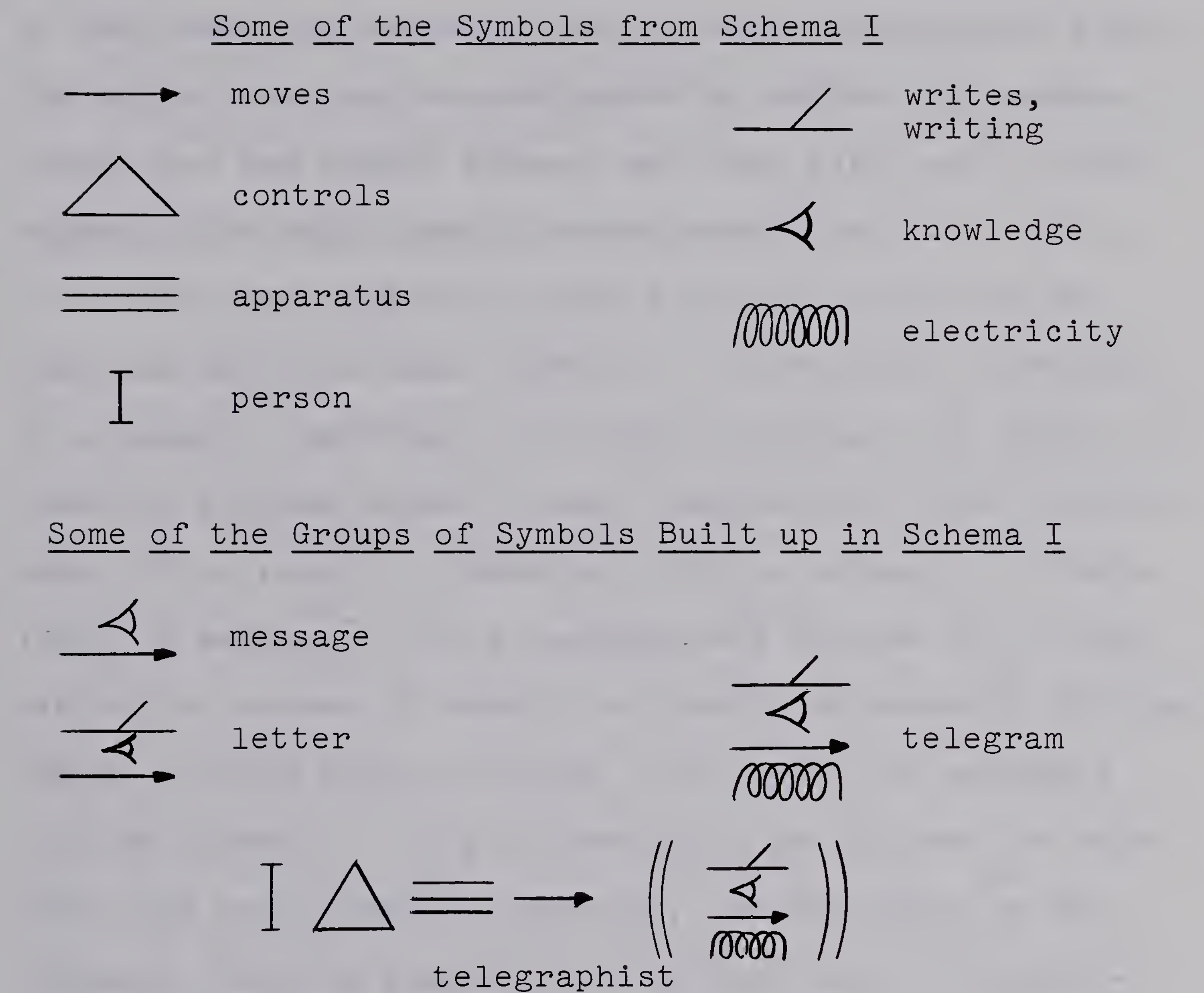
[185] *Ibid.*, pp. 135-138.



Note: acts as a frequentative, i.e., whatever is inside is taken as occurring over and over again.

FIGURE 1

SAMPLES FROM SKEMP'S SCHEMA I*

*R.R. Skemp, "The Need for a Schematic Learning Theory," British Journal of Educational Psychology, 32:136, 1962.

material assimilable to Schema II (rote learning for Group I; schematic learning for Group II). That is to say, in each case the students had to learn material for which the schema they had learned would be helpful in assimilating the new symbol groups, and they also had to learn material for which their learned schema was of no help. In neither case did the students need to associate new meanings with the basic symbols. To arrive at a measure of schematic learning, the scores of Group I on recall of material fitting Schema I were combined with the scores of Group II on recall of material fitting Schema II. Similarly, a measure of rote learning was arrived at by combining the scores of Group I on recall of material fitting Schema II with those of Group II on recall of material fitting Schema I. By thus combining the scores for schematic and non-schematic learning, the aid given by the schemata could be quantified. In this way, the experimental design allowed for any differences between the groups of students and differences in difficulty of the final tasks.[186]

The results from the experiment showed that the subjects forgot more of the rote-learned material in a day

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[186] Ibid., p. 138.

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than was forgotten of the schematically learned material over a period of four weeks. Furthermore:

> Schematic learning resulted in twice the number of symbols being recalled immediately after learning, and seven times as many after four weeks. It gave both faster learning and better retention.[187]

Skemp's Anxiety Studies and Related Research

Another facet of Skemp's investigations has been his assessment of the effects of anxiety on the functioning of the reflective processes. He designed an experiment to test the hypothesis that boys showing low mathematical achievement would exhibit greater anxiety and poorer performance at reflective processes than boys of higher achievement in mathematics. The experiment involved a group of boys whose mathematics performance was poor in comparison with their performance in other subjects and a group of boys who performed well in mathematics as compared with other subjects. The members of the groups were not all from the same forms, but pairs of boys of low and high mathematics achievement were chosen from the same forms. Anxiety was assessed in terms of a modification of Jung's word association test consisting of sixty-nine words of which fifteen had possible mathematical significance.

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[187] Ibid., p. 139.

The word list was given before and after an administration of Skemp's SK5 test. The criterion for identifying an emotionally toned word was that a different association was given in the two presentations. The subjects were told that the word association test was a measure of how quickly they could think of words.[188]

The results of the experiment are displayed in Table IV on page 163. It can be seen that the "low mathematics" group gave fewer (but not significantly fewer) repetitions of the same response word, had a greater mean response time, and scored lower on the test of operations manipulation (SK5 B). The six selected words referred to in Table IV were mathematically oriented words chosen as the best discriminators between the groups. The large number of neutral words in the test tended to mask the differences in mean response times between the groups. The low mathematics group was found to be slower and less accurate at manipulating operations and was found to be more anxious. The differences that were found between the groups were all in the predicted direction.[189]

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[188]Skemp, "An Investigation into Difficulties in Learning Mathematics by Children of Good General Intelligence," pp. 168-169.

[189]Ibid., pp. 171-173.

TABLE IV

MEAN SCORES OBTAINED BY THE HIGH MATHEMATICS ACHIEVEMENT (HM) GROUP AND THE LOW MATHEMATICS ACHIEVEMENT GROUP (LM) ON THE WORD ASSOCIATION TEST, SK5 A, AND SK 5 B*

Code:
- P -- mean number of words repeated on word association test
- Q -- mean response time in seconds for word association test
- Q'-- mean response time in seconds for six selected words from the word association test
- T -- mean total time in seconds to complete SK5 B

Group	N	P	Q	Q'	SK5 A	SK5 B	T
HM	10	40.5	2.4	2.2	39	25.7	760
LM	10	31.5	3.6	4.6	34.7	17.4	1024
Significance Level of Difference		NS	.05	.05	NS	.05	NS

*R. R. Skemp, "An Investigation into Difficulties in Learning Mathematics by Children of Good General Intelligence" (unpublished Doctoral Dissertation, The University of Manchester, 1958), p. 170.

Since the low mathematics students were better than the high mathematics students in verbal subjects but worse in mathematics, the two groups were considered to be approximately matched in general ability. Consequently, the significantly lower mean score on SK5 B obtained by the low mathematics group tends to support the contention that the SK5 B test measures something different from, or perhaps a single component of, general ability.[190]

In yet another study, Skemp attempted to induce anxiety-based mental blockage. The subjects were two groups of fourteen to fifteen-year-old grammar school boys chosen as being representative of good and poor mathematics students, respectively. The subjects were given duplicated sheets which displayed numbers in groups of three, with twelve such groups to a line and twenty-one lines on a page. The three main tasks given were to cross out the least, the intermediate, and the greatest figure, respectively, in each group. The very simple task of crossing out the first number in each group was also introduced in one part of the experiment. The subjects were instructed to cross out a figure for every click of a metronome and to begin a new line every time the experimenter called "next," even if

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[190] Ibid., p. 171.

they had fallen behind. After preliminary practice at slow speed, the metronome was set to click once every second, and the subjects were instructed to cross out least, intermediate, and greatest numbers in consecutive groups for every line on the first sheet. On the second sheet, the subjects were asked to cross out the first in each group on alternate lines while following the procedure of the first sheet on the other lines. A third sheet was completed in the same way as the first. The first sheet was designed to measure blockage resulting from the anxiety induced by rapid pacing and alternating decision tasks. The intention with the second sheet was to reduce the cumulative effect of anxiety by interpolating the simple task, and the third sheet was introduced to provide a control on any practice effect.[191]

Both groups, good and poor mathematics students, showed progressive decrement in the correct number of responses made in a given time on the first and third sheets. Since this decrement was absent from performance on the second sheet, support was gained for the hypothesis that mental blockage would be produced by the effects of cumulative anxiety. Since the task involved reflective processes,

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[191] Ibid., pp. 178-180.

switching response criteria for each successive item of the task, the view that reflective processes are susceptible to anxiety-based mental blockage also seemed to receive support.

Many other experiments have been conducted to investigate the effect of anxiety on performance, but detailed descriptions of the procedures used are not considered relevant in the present context. Instead, a selection of brief citations of general findings is presented.

In one experiment[192] it was found that low anxiety university freshmen, whose anxiety level was determined from an administration of Sarason's Test Anxiety Scale (TAS), produced significantly more correct responses to mental arithmetic tasks than high anxiety students. In another study[193] it was found that the most anxious subjects did as well as others on questions requiring specific information, but relatively poorly when analysis, application,

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[192] B. W. Harleston, "Test Anxiety and Performance in Problem Solving Situations," Journal of Personality, 30: 557-573, December, 1962.

[193] Benjamin S. Bloom, "Personality Variables and Classroom Performance," Journal of the National Association of Deans of Women, 16:159-164, June, 1953, cited by Donovan A. Johnson, "Implications of Research in the Psychology of Learning for Science and Mathematics Teaching," Review of Educational Research, 27:404, October, 1957.

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or synthesis was required. Furthermore, rigid subjects were found to be capable of handling highly structured, familiar material but to be unable to interrelate such material or apply it to new situations. Mandler and Sarason[194] found that a group of Yale freshmen with high anxiety scores on the TAS scored significantly lower on a mathematics aptitude test than a group of low anxiety freshmen did. I. G. Sarason[195] found that under threat conditions students with high TAS scores performed at a much lower level on a difficult anagram task than did medium and high anxiety students. However, there is a body of research that suggests that very low anxiety levels may also adversely affect performance. One reviewer of anxiety research has cited evidence to support the postulation of a possible inverted U-shaped distribution of the effects of anxiety upon learning--moderate anxiety being the most conducive to efficient learning and extremely low or high anxiety having a deleterious effect.[196] Summarizing the major points suggested by the present state of

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[194] S. B. Sarason and G. Mandler, "Some Correlates of Test Anxiety," Journal of Abnormal and Social Psychology, 47:813-815, 1952.

[195] I. G. Sarason, "The Effects of Anxiety and Threat on the Solution of a Difficult Task," Journal of Abnormal and Social Psychology, 62:165-168, 1961.

[196] Horace B. Reed, Jr., "Anxiety: the Ambivalent Variable," Harvard Educational Review, 30:143-144, Spring, 1960.

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research on the relationships between anxiety and learning, the reviewer wrote:

> a. Severe anxiety tends to depress learning.
> b. A mild degree of anxiety may function in a positive manner for some forms of learning.
> c. A very low level of anxiety may depress learning.
> d. Certain learnings may be independent of the anxiety variable.
> e. The nature of the learning criterion may account for variations in the effects of anxiety.
> f. The effects of anxiety may vary in relation to the strength of other antecedent variables present.
> g. No one theory explaining human behavior would appear to be sufficiently advanced to account for the variety of phenomena being reported.[197]

It is not surprising that the same reviewer has referred to anxiety as the ambivalent variable.

Prediction Studies

Optimum prediction of ninth grade success in mathematics for 192 junior high school students was obtained in one study[198] from a multiple regression prediction equation involving eighth grade mathematics marks and seventh grade *Iowa Every-Pupil Tests of Basic Skills* arithmetic scores. The multiple correlation of the optimally weighted sum with ninth grade algebra scores

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[197] *Ibid.*, pp. 152-153.

[198] W. E. Barnes and J. W. Asher, "Predicting Students' Success in First-year Algebra," *The Mathematics Teacher*, 55:651, December, 1962.

was 0.6245. The eighth grade mathematics marks and the seventh grade arithmetic scores correlated 0.588 and 0.546, respectively, with the ninth grade algebra scores. IQ scores, grade seven and eight reading scores, and Orleans Algebra Prognosis Test scores were found not to produce a significant practical increase in predictive efficiency.

In another study,[199] four aptitude batteries were administered to six classes of ninth grade beginning algebra students at the start of or prior to the course. The batteries were the Iowa Algebra Aptitude Test (IAAT), the Orleans Algebra Prognosis Test (OAPT), the SRA Primary Mental Abilities tests, and the Differential Aptitudes Tests, Form A (DAT). The correlations of these tests with the final algebra test scores were: IAAT, 0.66; OAPT, 0.67; SRA Verbal, 0.51; SRA Space, 0.33; SRA Reasoning, 0.49; SRA Number, 0.35; SRA Word Fluency, 0.37; DAT VR, 0.56; DAT NA, 0.63; DAT AR, 0.53; DAT SR, 0.47; DAT Mechanical Reasoning, 0.25; DAT Clerical Speed and Accuracy, 0.22; DAT Spelling, 0.53; DAT Sentences, 0.56; DAT VR+NA,

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[199]H. G. Osburn and R. S. Melton, "Prediction of Proficiency in a Modern and Traditional Course in Beginning Algebra," Educational and Psychological Measurement, 23:277-287, 1963.

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0.68. It is of special interest to note that the sum of the DAT Verbal Reasoning and Numerical Ability scores had a predictive validity equal to that of the Iowa Algebra Aptitude Test and the Orleans Algebra Prognosis Test, considering that the latter are specifically designed to predict proficiency in algebra.

Guilford, Hoepfner, and Peterson[200] conducted a study in which the California Test of Mental Maturity (CTMM), the Differential Aptitudes Tests (DAT), the Iowa Tests of Basic Skills (IOWA), and a battery of thirty-two "structure of intellect" factor tests were administered to 600 ninth grade mathematics students. The students were enrolled in four types of mathematics courses: Basic Mathematics, Non-college Algebra, Regular Algebra, and Accelerated Algebra. A preliminary factor analysis produced thirteen factor scores from twenty-five of the "structure of intellect" factor tests. Several groupings of tests were made and multiple correlations were calculated between these groupings of scores and mathematics achievement test scores. The groupings were: the nine

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[200]J. P. Guilford, R. Hoepfner, and H. Peterson, "Predicting Achievement in Ninth-grade Mathematics from Measures of Intellectual Aptitude Factors," Educational and Psychological Measurement 25:659-682, Autumn, 1965.

[illegible]

standard tests (CTMM, IOWA, and DAT batteries); two CTMM tests; three IOWA tests; four DAT tests; the seven factor tests not included in the factor analysis; the thirteen factor scores; and twenty factor predictors (the seven factor tests and the thirteen factor scores). The multiple correlation coefficients calculated are displayed in Table V on page 172.

It can be seen from the table that the combination of the four DAT tests is about as good a predictor of success in grade nine mathematics as the larger batteries in the experiment. The investigators hypothesized that the exceptional success of the DAT scores (VR, NA, AR, and Clerical Speed and Accuracy) could be attributed to the emphasis on "memory for symbolic implication" (numerical facility), the MSI factor, in the NA test. The MSI factor has been identified as a strong predictor of algebraic success.[201]

Stepwise multiple regression analyses were also carried out in which only those predictors were retained that made statistically significant (0.10 level) contributions to the prediction. The part of the results that is most relevant in the present context is that the best

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[201]*Ibid.*, pp. 662-675.

TABLE V

MULTIPLE PREDICTION OF MATHEMATICAL-ACHIEVEMENT SCORES FROM WEIGHTED COMPOSITES OF STANDARD TESTS AND OF FACTOR TESTS*

Prediction Battery	Basic Mathematics	Non-college Algebra	Regular Algebra	Accelerated Algebra
9 standard tests	.60	.53	.22	.74
2 CTMM tests	.34	.40	.18	.37
3 IOWA tests	.53	.31	.20	.62
4 DAT tests	.57	.53	.24	.70
7 factor tests	.42	.56	.27	.51
13 factor scores	.46	.45	.39	.75
20 factor predictors	.48	.54	.38	.74

*J. P. Guilford, R. Hoepfner, and H. Peterson, "Predicting Achievement in Ninth-grade Mathematics from Measures of Intellectual Aptitude Factors," Educational and Psychological Measurement, 25:672, Autumn, 1965.

prediction from standard test scores involved IOWA Reading Comprehension, DAT NA, DAT AR, and DAT Clerical Speed and Accuracy in predicting the final grades of the Accelerated Algebra students. The multiple correlation coefficient for that prediction was 0.76. The next best prediction produced a multiple correlation of 0.65 between an optimum weighting of the same tests and the achievement scores of the General Mathematics students.[202]

Factor Analysis Studies

On the basis of the results from a factorial analysis of the scores obtained by 136 twelve to fourteen-year-olds on a battery of tests including a group intelligence test, a geometry test, and nineteen tests of mathematical ability, M. Hamza[203] concluded that the most important factor in mathematics ability is the general intellective factor, which contributed by far the greatest part of the variance in his study [Figures were not quoted.]. He also found that "visual imagery" and "number" factors differentiated sharply between students retarded in mathematics and those

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[202] *Ibid.*, p. 676.

[203] Mukhtar Hamza, "Retardation in Mathematics Amongst Grammar School Pupils," *British Journal of Educational Psychology*, 22:189-195, November, 1952.

achieving passing grades.

J. Wrigley's review of the literature on factor analysis found general agreement that the most important factor necessary for success in mathematics is a general intellective factor, "g." Other, less important factors isolated have been visual imagery, number, and classification factors. Having thus reviewed the literature, Wrigley designed and carried out an experiment to try to find out if a group factor of mathematical ability over and above general ability could be isolated. He constructed mathematics achievement tests in algebra, geometry, mechanical arithmetic, and problem arithmetic. In addition, he assembled a battery of standard tests of verbal intelligence, reading comprehension, vocabulary, grammar, addition, subtraction, multiplication, division, and spatial reasoning. The mathematics achievement tests and the standard test battery were administered to 622 grammar and technical school boys ranging in age from thirteen to sixteen years.[204]

By orthogonal rotation of the axes obtained from a centroid factor analysis, Wrigley was able to establish a group factor pattern consisting of a general or basic

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[204]Jack Wrigley, "The Factorial Nature of Ability in Elementary Mathematics," British Journal of Educational Psychology, 28:68-94, 1958.

factor and four group factors: verbal, mathematical, spatial, and numerical. There was considerable overlap between the mathematical and numerical factors. Wrigley interpreted his findings to mean that although the first requisite for success in mathematics is good general intelligence, there is a specific mathematically oriented type of ability that could likely be nurtured by a unified approach in the teaching of the various branches of mathematics.[205]

Sister Canisia[206] administered a battery of thirty-six tests related to the measurement of mathematical ability to 160 girls in the eleventh grade of a private secondary school. Five of the tests were reference tests for a number factor, seven were reference tests for reasoning factors, two were reference tests for verbal factors, two were external reference criteria for mathematical ability, and one was a reference for a space factor. The rest of the tests were designed to explore hypotheses about the nature of mathematical ability. Twelve centroid factors were extracted from the thirty-six variable correlation

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[205] _Ibid_., pp. 74-77.

[206] Sister M. Canisia, "Mathematical Ability as Related to Reasoning and Use of Symbols," _Educational and Psychological Measurement_, 22:105-127, Spring, 1962.

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matrix, and rotation to simple structure in terms of oblique factors was carried out. The factors isolated were interpreted as being related to number, flexibility of closure, formal logic (ordering, scaling, organizing), manipulation of quantitative relationships, formal mathematics training, guessing, verbal meanings, eduction of correlates, eduction of abstract relationships, and word fluency. The nature of several factors suggested that mathematical processes are mainly processes of eduction, organization, and manipulation of relations. Four of the factors seemed to invoke the perception of relations and the use of this knowledge in the solution of problems. Such an interpretation appears to agree very well with Skemp's description of the reflective processes required for successful problem solving in mathematics.

V. CHAPTER SUMMARY

The review of the literature presented in this chapter has served: to illustrate how classical learning theories (conditioning, reinforcement, Gestalt, and field theories) have been inadequate to account for higher order mental processes such as those required in the learning of mathematics; to describe Piaget's theory of cognitive growth as a more adequate frame of reference for

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characterizing mathematics learning; to show how the theories of Bruner, Dienes, and Skemp are derived from a Piagetian point of view; and to marshall theoretical and empirical support for Skemp's theory of mathematics learning. In the succeeding paragraphs an attempt is made to summarize the discussions found in the chapter.

The classical learning theories appear to be inadequate for explaining classroom learning either through failure to account adequately for purposive behaviour or through failure to take cognizance of the effects of previous learning on present learning, or for both reasons. It has been found that useful implications for education can be derived from modifications of the classical theories, but such implications are largely restricted to learning through drill and to accounting for the effects of punishment and reward on mental processes.

Piaget's theory has been described as a fruitful alternative to classical learning theory. Such constructs as schemata, assimilation, accommodation, operations, reversibility, and equilibration, which are found in the theory, appear to be adequate to account for the effects of past learning and purposive behaviour. Piaget's insightful descriptions of cognitive development and the factors influencing such development have been found to be replete

with implications for mathematics education. For example: concepts should be built from appropriate "concrete" experiences; students can be led to create their own notions of the structure of a subject if they are given a rich experiential background; and flexible thinking is encouraged by bringing out the relationships between inverse operations and by giving students opportunities to approach problems from several points of view and various methods of attack.

Bruner's descriptions of concept development have been found insightful and applicable to classroom learning. The notions of Bruner and Dienes regarding recurring learning cycles which carry the learner to higher and higher levels of abstraction and generalization have been described as forming a useful interpretation and extension of Piaget's theory of cognitive growth. Dienes' ingenious concrete embodiments of mathematical concepts and operations have been viewed as important vehicles for providing mathematically rich classroom experiences.

Skemp's three-part theory of mathematics learning has been described in detail and support has been found for its main ideas by reference to the theories of Piaget, Bruner, and Dienes. The applications of concept formation notions to mathematical learning situations, as described by Skemp, have been found illuminating, especially with

regard to the question of useful mathematical definitions. Skemp's discussion of schematic learning has pointed to the necessity of providing opportunities for students to build adequate, meaningful, operational thinking structures. The notions of sensori-motor intelligence and reflective intelligence have been found to parallel major ideas in the theories of Piaget, Bruner, and Dienes and to provide useful insights into appropriate teaching procedures and learning activities. Skemp's discussion of anxiety and its effects on cognitive processes has provided warnings against the dangers that may arise in the emotional climate of the classroom. (More detailed discussion of implications from Skemp's theory is presented in the concluding chapter of this report.)

Finally, research studies have been reviewed in the present chapter to provide: further insights into applications of Piaget's theory at the secondary school level, empirical support for Skemp's theory, and support for some of the design decisions made in connection with the study presently being reported.

CHAPTER III

THE EXPERIMENTAL DESIGN AND STATISTICAL PROCEDURES

Is the presence of reflective intelligence a factor of enough importance for success in mathematics that the addition of measures of reflective intelligence to measures of general intelligence significantly improves the prediction of one's performance in mathematics? What effects on the relationships between measured reflective intelligence and mathematics performance result from taking into account student anxiety toward testing situations? Are there any significant differences among the mean levels of reflective intelligence exhibited by boys or girls in age categories from ten to sixteen? A major purpose of the study presently being reported was to produce evidence that might provide a basis for answering these questions.

The present chapter contains a detailed description of the experimental design and the statistical procedures used. Included are a discussion of the rationale underlying the choice of the measuring instruments used, technical descriptions of the measuring instruments, a discussion of the sampling and testing procedures used, a listing of the null hypotheses tested, and a discussion

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of the bases upon which the various statistical decisions were made.

I. THE EXPERIMENTAL DESIGN

Rationale Underlying the Choice of Measuring Instruments

As has been noted previously (pages 173 to 176), factor analytic studies have found a close connection between mathematical ability and general, numerical, and spatial ability. Since a central purpose of the study being reported was to determine whether or not reflective intelligence is an important prerequisite for mathematics learning, an attempt was made to assess its importance in predicting mathematical performance relative to measures of general, numerical, and spatial abilities, considering that the significance of the latter three has been demonstrated. Furthermore, since Skemp has found evidence to suggest that anxiety may block reflective thinking, this possibility was allowed for within the framework of the study. Consequently, measures of general ability, numerical ability, spatial ability, reflective intelligence, anxiety, and mathematical performance were included in the test battery used in the major part of the investigation.

The decision to use the *Differential Aptitude Tests*

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(DAT) battery was based on reports that it is the best available battery for measuring intellectual abilities[1] and that the inter-correlations of the tests in the DAT battery are low, indicating that the abilities measured do not overlap to any appreciable extent. Furthermore, coefficients of correlation of the order of 0.75 between the DAT _Numerical Ability_ (NA) test and a standardized arithmetic achievement test (grade eight level) have been reported.[2] A correlation of 0.85 between grade twelve College Entrance Examination Board (CEEB) _Scholastic Aptitude Test_ (SAT) -- Mathematics scores and scores from a prediction equation employing grade ten DAT _Numerical Ability_ (NA), _Verbal Reasoning_ (VR), and _Space Relations_ (SR) scores has also been reported.[3] The CEEB SAT -- Mathematics test is considered to be one of the best tests available for estimating a student's college level mathematical comprehension.[4] Reported

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[1]John B. Carroll, Reviewing the DAT Battery in _The Fifth Mental Measurements Yearbook_, O. K. Buros, editor (Highland Park, N.J.: The Gryphon Press, 1959), p. 673.

[2]G. K. Bennett, H. G. Seashore, and A. G. Wesman, _Differential Aptitude Tests Manual_ (third edition; New York: The Psychological Corporation, 1959), p. 74.

[3]_Ibid._, p. 84.

[4]Wayne S. Zimmerman, Reviewing the SAT Test in _The Sixth Mental Measurements Yearbook_, O. K. Buros, editor (Highland Park, N. J.: The Gryphon Press, 1965), p. 708.

correlations between grade twelve _Essential High School Content Battery_ -- Mathematics and DAT test scores have been: for VR, 0.65; for NA, 0.66; for _Abstract Reasoning_ (AR), 0.46; and for SR, 0.40.[5] In addition (as reported on pages 168 to 173), DAT scores have been found to be good predictors of ninth grade mathematical achievement. Considering these reports, which support the contention that the above-mentioned DAT ability measures are closely related to mathematical and arithmetical performance, and the claim that the DAT VR and NA tests together give a good measure of general mental ability while the DAT AR test provides a non-verbal measure of reasoning ability, another aspect of general intelligence,[6] the _Verbal Reasoning_ (VR), _Numerical Ability_ (NA), _Space Relations_ (SR), and _Abstract Reasoning_ (AR) tests from the DAT, Form L, battery were chosen to give measures encompassing general, numerical, and spatial abilities.

The entire study was based on Skemp's definition of reflective intelligence, so his tests of _Concept Formation_ (SK4, Part I), _Reflective Action with Concepts_ (SK4, Part II), _Operation Formation_ (SK6, Part I), and _Reflective Action with Operations_ (SK6, Part II) were used

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[5]Bennett, Seashore, and Wesman, _op. cit._, p. 56

[6]_Ibid._, p. 5

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to arrive at measures of reflective intelligence.

The product of a continuing long-term study of school children's anxiety toward test situations is Sarason's Test Anxiety Scale for Children (TASC). This test seemed admirably suited to the design of the proposed study and its validity is attested to by the results of numerous research studies. The scale can be administered in a relatively short time, and it was given immediately following administration of the DAT Space Relations test. The TASC was chosen partly because of its specific emphasis on test anxiety rather than on the more general anxieties and personality correlates which other available anxiety scales emphasize.

A test developed in a previous study[7] and called the Special Mathematical Understandings (SMU-II) test was used as a measure of mathematical performance because it had been specifically designed to measure the kinds of grade eight level mathematical understandings that are currently recognized as essential to continuing mathematical success. The test measures student performance in working with basic mathematical ideas rather than measuring powers of recall or computation.

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[7]D. B. Harrison, "An Analysis of the Effectiveness of Three Mathematics Programs at the Grade Eight Level" (unpublished Master's Thesis, The University of Alberta, Edmonton, 1964).

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In what follows, technical descriptions of the tests used in the study are given.

The Measuring Instruments

1. Verbal Reasoning (VR) test from the Differential Aptitude Tests (DAT), Form L, battery. The VR test is a measure of ability to understand concepts framed in words. It consists of 50 items cast in the form of verbal analogies. The vocabulary is relatively simple and the content is familiar. Item difficulty and complexity are functions of the reasoning process required. A Spearman-Brown reliability coefficient of 0.90 has been calculated for the VR test.[8] Extensive predictive validity studies have been carried out with each of the tests in the DAT battery, and literally hundreds of validity coefficients are quoted. Reviewers have not attempted to encapsulate the wealth of technical information available on the battery, but their assessment of the quality of the tests is unmistakable. For example, one reviewer has stated:

> The authors have done such a thorough and technically satisfactory job that a reviewer finds it hard to make himself appear sufficiently critical. With one or two possible exceptions, the tests are excellent in format, item construction, standardization,

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[8] Bennet, Seashore, and Wesman, op. cit., pp. 6, 66.

> validation, and just about every other aspect which is regarded as important in the testing fraternity.
> . . . .
>
> At the present time, it can be said that considering the tests themselves and all the supporting data, the DAT constitutes the best available foundation battery for measuring the chief intellectual abilities and learned skills which one needs to take account of in high school counseling.[9]

The DAT tests are designed to measure the abilities of boys and girls in grades eight through twelve. Administration time for the VR test: 40 minutes.

2. Numerical Ability (NA) test from the DAT, Form L, battery. There are 40 multiple choice items in the NA test and each includes a "none of these" option to discourage answer estimation techniques. The items are designed to test understanding of numerical relationships and facility in handling numerical concepts. To avoid the language element of the usual "arithmetic reasoning" problem (a verbal problem), all of the items are cast in an "arithmetic computation" form. Though some of the items test only for skill in numerical processes, many of the items are, as problems, fully as complex as items usually framed in verbal terms.[10] A

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[9] Carroll, *op. cit.*, pp. 672-673.

[10] Bennett, Seashore, and Wesman, *op. cit.*, p. 6.

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mean Spearman-Brown reliability coefficient of about 0.90 has been calculated for the NA test.[11] Administration time for the NA test: 40 minutes; for the VR and NA tests together: 75 minutes.

3. Space Relations (SR) test from the DAT, Form L, battery. The SR test consists of 60 items of the "unfolded paper boxes" type, which require the student to manipulate objects mentally in three dimensional space. The items employed combine visualization of how an object would appear if rotated in various ways with visualization of a constructed object from a given pattern, two factors considered important in ability to think in spatial terms.[12] A mean Spearman-Brown reliability coefficient of the order of 0.93 has been calculated for the SR test.[13] Administration time for the SR test: 35 minutes.

4. Abstract Reasoning (AR) test from the DAT, Form L, battery. The AR test is intended as a non-verbal measure of a student's reasoning ability to supplement the general intelligence aspects of the VR and NA tests. It involves the ability to perceive relation-

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[11] Ibid., p. 66. [12] Ibid., p. 7.

[13] Ibid., p. 66.

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ships in abstract figure patterns. A series of figures is presented in each problem and the subject is required to perceive an operating principle in the changing patterns in order to designate which of a number of choices of diagrams would follow next in the series. There are 50 items in the test.[14] A mean Spearman-Brown reliability coefficient of about 0.90 has been calculated for the AR test.[15] Administration time for the AR test: 35 minutes.

5. Skemp's Test of Concept Formation (SK4(1)). This test requires pupils to identify whether each of three test examples are exemplars or non-exemplars of a certain geometric concept after having been given three exemplars and three non-exemplars. The criterion of whether or not the student has formed the concept is if he can use it, not if he can verbalize it.[16] A detailed description of SK4(1) (also referred to as SK4, Part I is given on pages 147 and 148 of the present report, and a reproduction of the test is included in the Appendix. Since it was intended to administer Skemp's tests to children as young as ten years of age, both parts of the

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[14] Ibid., p. 7. [15] Ibid., p. 66.

[16] R. R. Skemp, "Reflective Intelligence and Mathematics," British Journal of Educational Psychology, 31:52, February, 1960.

SK4 test were drawn so that the figures were relatively large and unambiguous and so that the students could record their answers in the test booklet. The intention was to minimize distractions for the younger students. The SK4(1) test was designed to be administered immediately prior to Skemp's Reflective Action with Concepts (SK4(2)) test and to make the student aware of the concepts with which he is asked to act reflectively in the SK4(2) test.

6. Skemp's Test of Reflective Action with Concepts (SK4(2)). This test involves logical multiplication: combining two concepts to form a new concept in which possession of both of the properties of the original concepts is used as the criterion. The test material gives three exemplars of the double property, one non-exemplar exhibiting only one of the properties, one exhibiting the other, and one exhibiting neither, in that order. The subject is told how the non-exemplars are arranged. To grasp and demonstrate the new concept the subject has to think reflectively, not only being aware of the properties of the original concepts but deliberately combining and separating them, to indicate which of three test examples is an exemplar or non-exemplar of

all test were given so that the figures were relatively large and unambiguous and so that the subjects could record their answers in the test booklet. The intention was to minimize distractions for the younger students. The (XXXIII) test was designed to be administered immediately after the Kemp's Multiplying Cubes with Concepts (XXXIV) test and to have the highest number of the concepts [illegible] identical to the (XXXIII) test.

7. Kemp's Test of Partitioning Cubes with Concepts (XXXV). This test involves logical multiplication: combining two concepts to form a new concept in which possession of both of the properties of the original concept is basic to the criterion. The test material gives three exemplars of the joint property, one non-exemplar exhibiting only one of the properties, one exhibiting the other, and one exhibiting neither, in each order. The subject is told only the non-exemplary and exemplary [illegible] group and demonstrated the card through the [illegible] had to think differently, not only being aware of the properties of the original concepts but intellectually combining and separating them, to indicate which of three test examples is an exemplar or non-exemplar of

the double property.[17] A detailed description of SK4 (2) (also referred to as SK4, Part II) is given on pages 148 and 149 of the present report, and a reproduction of the test is included in the Appendix. On the basis of administration of the SK4(2) test to 138 subjects, Skemp has calculated a reliability coefficient of 0.76 (correlation of odd-even items, corrected by Spearman-Brown formula to the full length of the test).[18]

7. Skemp's Test of Operations Formation (SK4(1)). This test requires the subject to operate on test figures using operations illustrated on a demonstration sheet by means of three examples. A detailed description of the original version of SK6(1), labelled SK5 A, is given on page 149 of the present report. The main changes made are that SK6(1) consists of ten operations rather than fifteen and that (presumably) items more difficult than those in SK5 A were included in SK6(1) to increase the formulative (and hence reflective) aspects of this part of the test.[19] Skemp had calculated an odd-even item reliability coefficient of 0.94 for the SK5 A test.[20]

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[17]Ibid., pp. 52-53. [18]Ibid., p. 53.

[19]R. R. Skemp, "An Investigation into Difficulties in Learning Mathematics by Children of Good General Intelligence" (unpublished Doctoral Dissertation, The University of Manchester, 1958), p. 203.

[20]Skemp, "Reflective Intelligence and Mathematics," p. 53.

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Since the revised version embodies minor changes in the items in the light of data obtained from SK5, but the main principles of the test remain unchanged, one could expect a reliability coefficient of similar order for SK6 (1).

8. Skemp's Test of Reflective Action With Operations (SK6(2)). In this test the subject is required to demonstrate "combining," "reversing," and "reversing and combining" operations on test figures.[21] Since to do this test, the subject must have discovered what the ten basic operations in the SK6(1) test are, after the administration of the SK6(1) test and prior to the SK6(2) test, the operations are explained to the subjects. A detailed description of the original version of SK6(2), labelled SK5 B, is given on pages 149 to 150 of the present report. As in SK5 B, there are five "combine," five "reverse," and five "reverse and combine" items in SK6(2). Skemp has calculated an odd-even item reliability coefficient of 0.95 for the SK5 B test,[22] and one could expect a coefficient of similar order for the SK6(2) test.

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[21] Ibid. [22] Ibid.

9. Sarason's Test Anxiety Scale for Children (TASC). The TASC test consists of thirty questions which are read to the class after it has been explained to the subjects that no one but the investigator will see their answers to the questions. It is explained that there are no "right" or "wrong" answers as the questions are about how students feel, and different people think and feel differently. Because of time limitations, only twenty selected TASC items were used in the present study (A listing of the items used is included in the Appendix.). Some examples of the test items are:

> 1. Do you worry when the teacher says that she is going to ask you some questions to find [out] how much you know?
> 21. Do you worry a lot while you are taking a test?
> 24. When you are taking a test, does the hand you write with shake a little?
> 28. When the teacher says that she is going to give the class a test, do you get a nervous or funny feeling?[23]

The subject is asked to respond "Yes" or "No" to each question.

This type of test is considered appropriate because, from Sarason's theoretical point of view, anxiety

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[23] S. B. Sarason and others, Anxiety in Elementary School Children (New York: John Wiley and Sons, Incorporated, 1960), pp. 87-89.

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is a <u>conscious</u> experience which can be communicated to another person and because the direct questioning procedure meets group administration and ease-of-scoring requirements.[24] In devising items for the TASC, Sarason and his colleagues were guided by requirements that a "Yes" answer to a question should constitute an admission of behaviour that is experienced as unpleasant, that there should be an element of anticipation of unpleasant consequences, that there should be questions involving bodily reactions to test situations, and that there should be a sampling of reactions to a variety of test-like situations.[25] Consequently, a high TASC score reflects anxiety-like reactions in a variety of test-like situations. The test has been found suitable for use with children in grades one through six, and the writer believes that, with some minor modifications in wording, it is appropriate at the grade eight level.

From responses of grade five students, test-retest and split-half reliability coefficients of 0.82 and 0.88 (corrected by Spearman-Brown formula), respectively, have been reported.[26] TASC validation studies that have been

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[24]*Ibid.*, p. 89. [25]*Ibid.*, p. 90.

[26]S. B. Sarason and others, "A Test Anxiety Scale for Children," *Child Development*, 29:108, 1958.

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carried out support the conclusion that test anxiety is a measurable and meaningful variable.[27] Such studies have also led to some interesting discussions about distinguishing between "the less intelligent" and the "lower test scoring" (highly anxious) individual in, say, an IQ test situation. Strong support has also been received for the general hypothesis that children who are anxious in terms of the TASC measure have greater difficulty with tests than their non-anxious peers.[28]

11. Special Mathematical Understandings Test (SMU-II). This test, designed[29] in 1964, contains fifty multiple choice items which measure grade eight student understandings in the areas of Measurement, Number Systems, Numeration, Geometry, and Problem Solving. In the test items, mathematical understandings are abstracted from special terminology and symbolism, and the student is frequently required to use his mathematical concepts in novel situations. The involvement of computational skill has been minimized. The SMU test is believed to have

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[27] Ibid., p. 111.

[28] Sarason and others, Anxiety in Elementary School Children, p. 188.

[29] Harrison, op. cit., pp. 75-81.

acceptable content validity as the concepts on which the items are based were taken largely from grade eight SMSG materials and as responsible persons in the field of mathematics education have thoroughly studied the items for weaknesses. Using responses to the test items from 604 grade eight students, a Kuder-Richardson (Formula 20) reliability index of 0.83 has been calculated for the SMU test.[30] Since the fifty item SMU test requires a 45 minute administration time and since, in many of the schools participating in the study, the class periods were only 43 minutes long, five of the original items were deleted. The resulting forty-five item test is referred to in what follows as the SMU-II test. Decisions about which items were to be deleted were based on information obtained from an item analysis of responses from 148 students with mathematics backgrounds similar to those of the students tested in the present study.

Sampling and Testing Procedures

Prior to the initiation of the main study, a trial administration of Skemp's tests to a class of grade eight students (not included in the main study) was carried out.

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[30] Ibid., p. 83.

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Student reactions to the test instructions and procedures were noted so that any necessary refinements could be made in anticipation of difficulties that might have arisen during testing in the main study.

For the main part of the study, six classes of grade eight students were chosen (by using a table of random numbers) from the set of grade eight classes in the Edmonton Public School System in the 1965-66 school year. Testing was initiated during the first week of May, 1966. The investigator personally administered the test battery in seven class periods, at the rate of two testing sessions a week, in the following sequence:

1. DAT _Verbal Reasoning_ (VR),
2. DAT _Numerical Ability_ (NA),
3. DAT _Abstract Reasoning_ (AR),
4. DAT _Space Relations_ (SR) and Sarason's _Test Anxiety Scale for Children_ (TASC),
5. Skemp's _Concept Formation_ (SK4(1)), and _Reflective Action with Concepts_ (SK4(2)) tests,
6. Skemp's _Operation Formation_ (SK6(1)) and _Reflective Action with Operations_ (SK6(2)) tests,
7. The _Special Mathematical Understandings Test_ (SMU-II).

The administration procedures used with Skemp's tests, the SMU-II test, and the TASC questionnaire are described in the Appendix. The DAT tests were administered according to the standard set of instructions that accompanies the battery. Complete sets of data were obtained for

131 grade eight students ranging in age from twelve to fifteen. The DAT VR+NA scores of the subjects ranged from 24 to 80 (from the 15th to the 99th percentile of the test publisher's norms).[31] The sample VR+NA median was in the 46-48 range as compared with a range of 37-39 for the test norm group. How the scores of grade eight Alberta students in general would compare with the test publisher's norms is a matter for conjecture, but it would appear that the sample as a whole evidenced somewhat better than average scholastic aptitude.

Whereas the testing described in the preceding paragraph formed the basis for the main part of the study, another phase of the study, related to making age and sex group comparisons with respect to measurable reflective intelligence, was based on data collected from the administration of Skemp's tests in two class periods to two classes of students at each of the grade five, six, seven, nine, and ten levels. For each grade, the classes were chosen from the appropriate set of classes in the Edmonton Public School System by means of a table of random numbers. The grade five, six, seven, and nine classes

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[31]G. K. Bennett, H. G. Seashore, and A. G. Wesman, Differential Aptitude Tests Manual (fourth edition; New York: The Psychological Corporation, 1966), p. "3-18."

wrote Skemp's tests in May and June, 1966, but a conflict with school final examinations necessitated the postponement of testing of the grade ten and eleven classes until September, 1966. Since Skemp's tests are not specifically school-subject oriented but rather focus on one's ability to think reflectively in abstract situations, it was assumed that the two month time lapse would have only minimal effects on the relative magnitudes of the mean scores on Skemp's tests. Scores on Skemp's tests from the first two grade eight classes that had been chosen for participation in the main part of the study were also used in the age level by sex comparisons phase of the study. Complete sets of data were obtained for 177 boys and 163 girls ranging in age from ten to sixteen.

The Null Hypotheses Tested

The data gathered from the testing program described on the preceding pages were used in statistical analyses designed to test the following null hypotheses:

Null Hypothesis 1. The efficiency of prediction of mathematics performance scores of grade eight students is not significantly improved by adding to a standardized battery of aptitude tests

a) a measure of concept formation skill,
b) a measure of ability to act reflectively on concepts,
c) a measure of operations formation skill,

[illegible]

d) a measure of ability to act reflectively on operations,
e) any variable that takes into account interactions among the various mathematics performance predictors.

Null Hypothesis 2. The efficiency of prediction of mathematics performance scores of grade eight students is not significantly improved by adding to a battery of aptitude tests and reflective intelligence tests
a) a measure of student anxiety toward test situations,
b) any variable that takes into account interactions among test anxiety scores and the other mathematics performance predictor scores.

Null Hypothesis 3. For school children in grades five through eleven, there are no significant differences among group mean scores obtained by students in age categories from ten to sixteen, or between boys' and girls' group mean scores, on tests of
a) concept formation,
b) reflective activity with concepts,
c) operations formation, or
d) reflective activity with operations,
and there are no significant interaction effects among the age level and sex categories.

II. THE STATISTICAL PROCEDURES

Each student who had written all of the tests that had been administered at his grade level was assigned an identification number coded by grade, class, and individual, and his ID number, test scores, sex code number, and age were punched on an IBM card in preparation for data processing on the University of Alberta's digital computer.

The procedures followed in analyzing the data are described in succeeding paragraphs. A listing of the scores obtained by the 422 students for whom complete sets of data had been collected is included in the Appendix.

The principal technique employed in making decisions with respect to Null Hypotheses 1 and 2, on which the assessment of the importance of reflective intelligence and test anxiety measures in predicting mathematics performance was based, was Efroymson's[32] method of multiple regression analysis that has come to be called "stepwise regression." As in most multiple regression procedures, "stepwise regression" enables one to arrive at a "best fit" prediction equation of the form

$$y = b_0 + b_1x_1 + b_2x_2 + \ldots + b_nx_n + e$$

in which

y is the dependent (criterion) variable,

$x_1, x_2, \ldots$ are the independent (predictor) variables,

$b_0, b_1, \ldots$ are the coefficients that produce the "best fit," and

e is the error term (difference between predicted

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[32] M. A. Efroymson, "Multiple Regression Analysis," Mathematical Methods for Digital Computers, A. Ralston and H. S. Wilf, editors (New York: John Wiley and Sons, Inc., 1960), pp. 191-203.

and actual values of the dependent variable). The "best fit" is defined by the set of coefficients, b_0, b_1, . . ., that makes the sum of the e^2 a minimum for a particular series of criterion values and predictor values obtained from a given sample. This is the familiar "least squares" method of determining a best fit of predicted values to actual values. The "stepwise regression" procedure produces a series of intermediate regression equations of the form

$$
\begin{aligned}
y &= b_0^{(1)} + b_1^{(1)}x_1 + e^{(1)} \\
y &= b_0^{(2)} + b_1^{(2)}x_1 + b_2^{(2)}x_2 + e^{(2)} \\
y &= b_0^{(3)} + b_1^{(3)}x_1 + b_2^{(3)}x_2 + b_3^{(3)}x_3 + e^{(3)} \\
&\vdots
\end{aligned}
$$

in which the variable added at each step is the one which makes the greatest improvement in "goodness of fit."[33] That is to say, at each step the variable added is the one that accounts for the greatest proportion of the remaining variance of the dependent (criterion) variable. Or, in other words, the variable added produces the greatest reduction in the sum of the squared error terms. New coefficients are determined at each step to produce the

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[33] *Ibid.*, pp. 191-192.

[illegible]

$$[illegible]$$

[illegible]

[illegible]

"best fit" in terms of the specific variables included in the prediction equation at that point.

The stepwise regression procedure has been programmed for data processing with the University of Alberta's digital computer,[34] and the program was made available to the investigator by the Division of Educational Research Services (DERS) of the Faculty of Education, University of Alberta. The stepwise regression program gives one the option of entering the predictor variables into the regression equation in stepwise order, in some pre-assigned order, or in a pre-assigned order for any number of variables and stepwise order for the rest. The generation of sums, products, powers, and other functions of the original variables is also possible, and these functions may be entered into the prediction equation.

The output of the stepwise regression program includes a listing of the means, variances, standard deviations, and the correlation coefficient matrix for the variables read in and generated. At each step of the

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[34]Department of Computing Science, "Multiple Regression" (A mimeographed description of the stepwise regression procedure programmed by Mrs. W. B. Payne; The University of Alberta, Edmonton, Alberta, October, 1964).

stepwise regression procedure, the following values are output:[35]

1. The regression equation coefficients, b_0, b_1, . . . , as estimated from the sample observations.

2. The standard error associated with each coefficient estimate.

3. A series of <u>t</u> ratios for testing the hypotheses that the coefficients in the prediction equation are not significantly different from zero. These <u>t</u> ratios are distributed with N - n - 1 degrees of freedom, where N is the number of sets of observations, and n is the number of predictor variables included in the regression equation.

4. The squared multiple correlation, R^2, between the criterion and the optimum weighting of the predictors.

5. The sum of squares of the criterion, referred to as the total sum of squares (SST).

6. The sum of squares of the predicted values of the criterion, referred to as the regression sum of squares (SSR).

7. The error sum of squares (SSE = SST - SSR).

8. The regression mean square (MSR) and the error mean square (MSE).

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[35] Ibid., pp. 2-6, 28.

9. An *F* ratio, F_R, distributed with n and N - n - 1 degrees of freedom, which, in effect, gives a test of the hypothesis that R^2 is not significantly different from zero. The value of F_R is calculated from

$$F_R = \frac{SSR/n}{SSE/(N - n - 1)} = \frac{MSR}{MSE} ,$$

which can be shown to be equivalent to the *F* ratio for testing the significance of R^2 that is given by Cooley and Lohnes,[36] namely,

$$F = \frac{R^2(N - n - 1)}{(1 - R^2)n} .$$

The equivalence of the two *F* ratios can be demonstrated by simple algebra by using the fact that

$$R^2 = \frac{SSR}{SST} ,$$

which is to say that R^2 is a measure of the proportion of the criterion variance that is accounted for by the predicted scores.[37]

10. The probability, P_R, of observing an *F* ratio greater than F_R.

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[36] William W. Cooley and Paul R. Lohnes, *Multivariate Procedures for the Behavioral Sciences* (New York: John Wiley and Sons, Inc., 1962), pp. 34-35.

[37] R. J. Hader and A. H. E. Grandage, "Simple and Multiple Regression Analyses," *Experimental Designs in Industry*, Victor Chew, editor (New York: John Wiley and Sons, Inc., 1958), p. 128.

11. A *t* ratio, t_n, distributed with N - n - 1 degrees of freedom, which gives a test of the hypothesis that the nth variable entered into the regression equation makes no significant contribution to the prediction. The *t* ratio is calculated from

$$t_n = \sqrt{\frac{SSR_n - SSR_{n-1}}{MSE_n}} \quad , n \geq 2, \text{ where}$$

SSR_n is the regression sum of squares with n predictors in the regression equation,

SSR_{n-1} is the regression sum of squares with the n - 1 predictors in the equation prior to the introduction of the nth, and

MSE_n is the error mean square for the prediction equation involving n predictors.

When the data from the present study were analyzed, a variable was retained in the prediction equation only if the value of t_n and of the appropriate regression coefficent *t* ratio exceeded the 0.01 level critical value, indicating that the increase in the squared multiple correlation was significantly different from zero and that the coefficient of the variable in question was significantly different from zero.

For cases in which it was desired to contrast two squared multiple correlations not in the step-by-step sequence, the following F ratio[38] was used to test the hypothesis that the squared multiple correlations were not significantly different from one another.

$$F = \frac{(R_1^2 - R_2^2)(N - n_1 - 1)}{(1 - R_1^2)(n_1 - n_2)} \quad \text{, where}$$

R_1 = the multiple correlation involving the greater number of predictor variables,

R_2 = the multiple correlation with one or more variables omitted,

n_1 = the number of predictor variables associated with R_1,

n_2 = the number of predictor variables associated with R_2,

F is distributed with $(n_1 - n_2)$ and $(N - n_1 - 1)$ degrees of freedom.

As has been mentioned, the squared multiple correlation gives a measure of the proportion of variance of the criterion that is accounted for by the predicted scores.

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[38] J. P. Guilford, Fundamental Statistics in Psychology and Education (4th Edition; New York: McGraw-Hill Book Company, 1965), p. 403.

[illegible]

[illegible]

$$[illegible]$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Similarly, the square of a simple correlation between two variables gives a measure of the proportion of the variance of one variable that can be predicted from, explained by, or attributed to the variance of the other variable. The way the stepwise regression procedure is carried out, the variable entered into the prediction equation first is simply the one that has the greatest absolute correlation with the criterion. Since it is possible that several variables might have correlations with the criterion that are not significantly different from one another and since whichever variable is entered into the prediction equation first markedly influences the <u>apparent</u> contributions of variables entered in succeeding steps, it was considered fruitful to investigate which correlations with the criterion were not significantly different from one another so that alternative choices for the first variable entered could be investigated. For this purpose, the test statistic used was

$$t = (r_{xz} - r_{yz})\sqrt{\frac{(N - 3)(1 + r_{xy})}{2(1 - r_{xy}^2 - r_{xz}^2 - r_{yz}^2 + 2r_{xy}r_{xz}r_{yz})}} ,$$

a t ratio distributed with N - 3 degrees of freedom. Hotelling had devised this test of the significance of the difference between two correlations, r_{xz} and r_{yz}, calculated from observations on the same set of individuals,

[illegible]

[illegible]

[illegible]

without making any assumptions about the distribution of the variables in the population

> . . . but with the limitation that generalization is only to a subpopulation of all possible samples for which X and Y have exactly the same set of values as those in the observed sample.[39]

In view of the remarks made by Cooley and Lohnes[40] in a footnote referring to Efroymson's stepwise regression procedure, the foregoing limitation is not considered as posing a serious added restriction.

In order to verify and extend some of Skemp's findings regarding partial correlations between his tests and a mathematics criterion, the following formulae[41] for calculating partial correlation coefficients were used.

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}}$$

$$r_{12\cdot34} = \frac{r_{12\cdot3} - r_{14.3}\, r_{24.3}}{\sqrt{(1 - r_{14.3}^2)(1 - r_{24.3}^2)}}$$

in which

$r_{12.3}$ is the correlation between variables 1 and 2 with the effects of variable 3 removed.

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[39] Helen M. Walker and Joseph Lev, Statistical Inference (New York: Henry Holt and Company, Inc., 1953), p. 257.

[40] Cooley and Lohnes, op. cit., p. 35.

[41] Guilford, op. cit., pp. 339-340.

[illegible]

r_{12} is the correlation between variables 1 and 2, and

$r_{12 \cdot 34}$ is the correlation between variables 1 and 2 with the effects of variables 3 and 4 removed.

To test the hypothesis that a correlation or partial correlation calculated from sample scores was not significantly different from zero, the t ratio[42] used was

$$t = \frac{r_{12 \cdot 34 \ldots m}}{\sqrt{\dfrac{1 - r^2_{12 \cdot 34 \ldots m}}{N - m}}} ,$$

which is distributed with N - m degrees of freedom.

Additional information about the relationships among the various predictor variables and the mathematics criterion was obtained by computing canonical correlations between the predictors and five subtest scores (Measurement, Number Systems, Numeration, Geometry, and Problem Solving) from the SMU-II test. The canonical correlation procedure produces weighting systems for two sets of variables to maximize the correlation between the weighted scores. A series of weighting systems, each independent

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[42] Quinn McNemar, Psychological Statistics (third edition; New York: John Wiley and Sons, Inc., 1962), p. 167.

[illegible]

of (orthogonal to) the preceding systems, is calculated and the significance of each of the canonical correlations is tested by using Bartlett's lambda test.[43] The patterns of weights that are determined by the procedure indicate which of the scores in the two sets are most closely related. A DERS canonical correlation program, whose computational procedures parallel those described by Anderson,[44] was employed to give insights into the relationships between the various predictors and the mathematics criterion subtest scores gathered in the study presently being reported.

Finally, in order to test the various parts of Null Hypothesis 3 (page 199), the scores of the subjects tested in grades five through eleven were grouped into categories according to the subject's age (ten, eleven, twelve, thirteen, fourteen, fifteen, or sixteen) and sex (boy, girl). The number of scores from each of the four tests that fell into each category is indicated in Table VI. A chi square test on the cell frequencies in Table VI indicated that they were not significantly different from being proportional to

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[43] Cooley and Lohnes, _op_. _cit_., p. 37.

[44] T. W. Anderson, _An Introduction to Multivariate Statistical Analysis_ (New York: John Wiley and Sons, Inc., 1958), pp. 288-306.

TABLE VI

CELL FREQUENCIES FOR AGE GROUP AND SEX GROUP CLASSIFICATIONS OF STUDENTS TESTED IN GRADES FIVE THROUGH ELEVEN

AGE	SEX		TOTAL
	BOY	GIRL	
10	11	19	30
11	29	30	59
12	34	20	54
13	25	25	50
14	22	24	46
15	20	14	34
16	36	31	67
TOTAL	177	163	340

row and column totals.[45] Since the numbers in the sample groups were not necessarily representative of population figures and since the cell frequencies could be considered proportional to row and column totals, two-way unweighted means analyses of variance were considered appropriate for testing the four parts of Null Hypothesis 3.[46]

An assumption underlying the analysis of variance method is that the error (within cells) variance is homogeneous. The tenability of this assumption for each analysis of variance carried out was confirmed by employing Cochran's test for homogeneity of variance as described by Winer.[47] The computational procedures for a two-way unweighted means analysis of variance (ANOVA) as described by Winer[48] had been programmed for use with the University of Alberta's computer, and this program was used to produce statistical tests for the four parts of Null Hypothesis 3.

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[45]The observed chi square value was 6.73 as compared with 0.01 and 0.10 level critical values of 16.81 and 10.64, respectively.

[46]B. J. Winer, Statistical Principles in Experimental Design (edition II; New York: McGraw-Hill Book Company, Inc., 1962), p. 224; and R. L. Anderson and T. A. Bancroft, Statistical Theory in Research (New York: McGraw-Hill Book Company, Inc., 1952), pp. 278-279.

[47]Winer, *op. cit.*, pp. 94-95. [48]*Ibid.*, pp. 241-244.

When the analysis of variance indicated significant differences among the group means, tests of the significance of the differences between pairs of group means were carried out as described by Winer.[49] The test statistic used was

$$t = \frac{\bar{A}_i - \bar{A}_j}{\sqrt{\dfrac{2MS_{w\cdot cell}}{n_h q}}} \quad , \text{ where}$$

t is distributed with $N - pq$ degrees of freedom,

$\bar{A}_i$ is the mean of means for row (column) i

$\bar{A}_j$ is the mean of means for row (column) j

$MS_{w\cdot cell}$ is the pooled within cell variance,

n_h is the harmonic mean of the cell frequencies,

p is the number of rows (columns),

q is the number of columns (rows).

Had it been necessary to investigate the contrasts among the cell means, the Newman-Keuls procedure[50] would have been employed.

As with each of the statistical tests described in this chapter, 0.01 level critical values were used in the statistical tests for assessing the significance of differences between means. Consequently, the statistical

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[49]<u>Ibid</u>., pp. 208, 223.

[50]<u>Ibid</u>., pp. 101-103.

[illegible]

$$[illegible]$$

[illegible]

[illegible]

[illegible]

decisions made were relatively conservative since the probability of indicating a significant difference as a result of sampling error rather than reflecting a true difference in the population was only 0.01.

The present chapter has consisted of a description of the experimental design and of the statistical procedures followed in analyzing the data collected in the study being reported. The following chapter presents the results of the analyses carried out.

CHAPTER IV

THE RESULTS OF THE INVESTIGATION

In this chapter the results of the investigation are presented in two parts. The first section contains the findings from the major part of the study, an attempt to assess Skemp's reflective intelligence measures as predictors of mathematics performance in relation to a standardized aptitudes battery and a test anxiety scale. The subjects for this phase of the study were 131 grade eight students. The second part of the chapter contains the findings from the administration of Skemp's tests to 340 students ranging in age from ten to sixteen. The purpose of the latter part of the study was to make an assessment of the levels of reflective intelligence exhibited by school children of various ages.

I. ASSESSMENT OF THE CONTRIBUTIONS OF REFLECTIVE INTELLIGENCE AND ANXIETY SCORES IN PREDICTING MATHEMATICS PERFORMANCE

Displayed in Table VII is the matrix of Pearson product-moment correlations among the scores obtained by 131 grade eight students on the *Verbal Reasoning* (VR),

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

TABLE VII

CORRELATIONS, MEANS, AND STANDARD DEVIATIONS FOR THE SCORES OF 131 GRADE EIGHT STUDENTS FROM A BATTERY OF ABILITY, REFLECTIVE INTELLIGENCE, ANXIETY, AND MATHEMATICS TESTS

TEST	VR	NA	AR	SR	SK4(1)	SK4(2)	SK6(1)	SK6(2)	TASC	SMU-II
VR	1.000	0.474	0.472	0.357	0.340	0.467	0.474	0.441	−0.294	0.537
NA		1.000	0.476	0.364	0.332	0.432	0.387	0.540	−0.303	0.568
AR			1.000	0.489	0.356	0.445	0.584	0.533	−0.229	0.482
SR				1.000	0.273	0.355	0.505	0.491	−0.222	0.334
SK4(1)					1.000	0.492	0.487	0.394	−0.177	0.362
SK4(2)						1.000	0.452	0.447	−0.085	0.394
SK6(1)							1.000	0.604	−0.261	0.478
SK6(2)								1.000	−0.162	0.491
TASC									1.000	−0.196
SMU-II										1.000
MEAN	24.18	23.13	33.34	27.65	10.27	15.01	19.54	14.42	10.18	14.14
STD. DEV.	7.46	6.19	7.58	9.70	3.74	4.51	5.46	7.66	4.02	5.13

Numerical Ability (NA), Abstract Reasoning (AR), Space Relations (SR), Concept Formation (SK4(1)), Reflective Action with Concepts (SK4(2)), Operations Formation (SK6(1)), Reflective Action with Operations (SK6(2)), Test Anxiety Scale for Children (TASC), and Special Mathematical Understandings (SMU-II) tests. For a sample consisting of 131 sets of observations, a correlation greater than or equal to 0.203 would be considered significantly different from zero (This figure was obtained by substituting $t_{.01}(129df) = 2.36$ in the t ratio formula given at the top of page 209.). The only variable that did not have a significant correlation with the criterion was TASC. In addition to the correlation matrix, Table VII contains the means and standard deviations of the ten variables. The data in Table VII formed the basis for most[1] of the statistical analyses carried out to test Null Hypothesis 1 and Null Hypothesis 2, which are here re-stated for reference.

> Null Hypothesis 1. The efficiency of prediction of mathematics performance scores of grade eight students is not significantly improved by adding to a standardized battery of aptitude tests
>
> a) a measure of concept formation skill,

---

[1]For the analyses that included sums and products of the original variables, additional correlation coefficients were involved. These are given in the Appendix.

[illegible]

b) a measure of ability to act reflectively on concepts,
c) a measure of operations formation skill,
d) a measure of ability to act reflectively on operations,
e) any variable that takes into account interactions among the various mathematics performance predictors.

Null Hypothesis 2. The efficiency of prediction of mathematics performance scores of grade eight students is not significantly improved by adding to a battery of aptitude tests and reflective intelligence tests

a) a measure of student anxiety toward test situations,
b) any variable that takes into account interactions among test anxiety scores and the other mathematics performance predictor scores.

Null Hypotheses 1a), 1b), 1c), 1d), and 2a) were tested by carrying out a stepwise regression analysis with the SMU-II variable as the criterion and the other nine variables as predictors. Table VIII lists the variables that made a significant (0.01 level) contribution to prediction of the SMU-II scores in the order that they were entered into the regression equation along with the corresponding multiple correlations (R) with the criterion, F ratios (F_R) for testing the significance of the squared multiple correlations (R^2), probabilities (P_R) of observing F ratios as large as F_R, the percentage of the criterion variance accounted for by the predicted scores,[2]

---

[2]Perhaps a more accurate wording here would be "percentage of variance shared by the criterion and predicted scores," but the phrasing employed is commonly used in educational circles and it is convenient.

b) a measure of ability to see relationships in concepts,
d) a measure of operational thinking ability,
f) a measure of ability to [illegible] synthesis on [illegible],
e) any variable that takes into account interactions among the variable [illegible] constituent predictors.

Null Hypothesis 2. The efficiency of prediction of mathematics performance scores of grade eight students is not significantly improved by adding to a battery of aptitude tests and intelligence tests
a) a measure of student anxiety toward test situations;
b) any variable that takes into account interactions among test anxiety scores and the other mathematics performance predictor scores.

Null hypotheses 1(a), 1(b), 1(c), and 1(e) were tested by carrying out a stepwise regression analysis with the SAT-II [illegible] as the criterion and the other nine variables as predictors. Table [illegible] lists the variables that made a significant (0.01 level) contribution to prediction of the SAT-II scores in the order that they were entered into the regression equation along with the corresponding multiple correlations (R) and the associated F values (F_R) for testing the significance of the observed multiple correlations (R), probabilities (P_R) of obtaining F ratios as large as F_R, the percentage of the criterion variance accounted for by the predicted scores.[2] 2

[2]Perhaps a more accurate wording here would be "percentage of variance shared by the criterion and predicted scores," but the [illegible] wording employed is commonly used in educational circles and is so employed.

TABLE VIII

SIGNIFICANT PREDICTORS OF SMU-II SCORES IN THE ORDER ENTERED DURING STEPWISE REGRESSION ANALYSIS

Step, n	Predictors Included in Regression Equation	Multiple Correlation, R	F_R	P_R	Percentage of Variance Accounted for	t_n
1	NA	0.5676	61.31	0.000	32.21	____
2	NA, VR	0.6439	45.32	0.000	41.46	4.49
3	NA, VR, SK6(1)	0.6696	34.41	0.000	44.84	2.79
4	NA, VR, SK6(1), AR	0.6745	26.29	0.000	45.49	1.23

Best Prediction Equation: $\hat{X}_{SMU-II} = {}^{-}1.084 + 0.298X_{NA} + 0.182X_{VR} + 0.201X_{SK6(1)}$

Coefficient t ratios: 4.69 3.31 2.79

$t_{.01}(127 \pm 1df) = 2.36$

_t_ ratios (t_n) for testing the significance of the contribution from the variable added at each step, the "best" prediction equation, and _t_ ratios for testing the significance of the weights (coefficients) for each variable in the best prediction equation. As can be seen from the table, the first variable entered into the regression equation was NA, followed by VR and SK6(1). The multiple correlation between the SMU-II scores and the composite predicted scores produced by the regression equation involving the NA, VR, and SK6(1) scores was 0.6696, indicating that 44.84 per cent of the variance of the criterion was accounted for by the predicted scores. The squared multiple correlation was significantly different from zero. However, as has been discussed previously (pages 207-208), the first variable chosen for entry into the regression equation is the one with the numerically greatest correlation with the criterion. As shown in Table VII, the correlations between the SMU-II scores and the VR, NA, AR, SK6(1), and SK6(2) scores were 0.537, 0.568, 0.482, 0.478, and 0.491, respectively. The _t_ ratio values calculated for comparisons between the NA correlation with the criterion and those of the VR, AR, SK6(1), and SK6(2) variables were 0.44, 1.20, 1.18, and 1.14, respectively. (Critical values: $t_{.01}$(128df) = 2.36, $t_{.05}$(128df) = 1.66.). Since

the five correlations were thus not significantly different from one another, there was no real reason why any of the five variables should not be entered into the regression equation first (other than an apparent numerical, but not statistically significant, difference). Since Skemp's tests were of central interest and since the correlation between the SK6(2) and SMU-II scores was numerically greater than those associated with the other scores from Skemp's tests, the SK6(2) variable was forced into the regression equation first--to find out what the nature of the resulting prediction equation would be. The findings are displayed in Table IX. The predicted scores from the regression equation involving SK6(2), NA, and VR had a multiple correlation (0.6606) with the criterion scores that was, for any practical purpose, not significantly different from that of the predicted scores from the NA, VR, and SK6(1) combination (0.6696). However, one could argue that the VR, NA, SK6(1) combination would be the more efficient, involving two forty minute tests and one ten minute test rather than three forty minute tests. In any event, the findings certainly support rejection of Null Hypotheses 1c) and 1d). It appears that the scores from the tests of operations formation and reflective activity with operations did make a significant contribution, over and above the verbal reason-

TABLE IX

SIGNIFICANT PREDICTORS OF SMU-II SCORES WHEN SK6(2) FORCED INTO REGRESSION EQUATION FIRST AND OTHER VARIABLES ENTERED IN STEPWISE ORDER

Step, n	Predictors Included in Regression Equation	Multiple Correlation, R	F_R	P_R	Percentage of Variance Accounted for	t_n
1	SK6(2)	0.4909	40.97	0.000	24.10	___
2	SK6(2), NA	0.6084	37.61	0.000	37.01	5.12
3	SK6(2), NA, VR	0.6606	32.78	0.000	43.64	3.86
4	SK6(2), NA, VR, SK6(1)	0.6733	26.12	0.000	45.33	1.98

Best Prediction Equation: $\hat{X}_{SMU\text{-}II} = 1.106 + 0.121X_{SK6(2)} + 0.271X_{NA} + 0.208X_{VR}$

Coefficient *t* ratios: 2.22 3.92 3.86

$t_{.01}(127 \pm 1df) = 2.36$

ing and numerical ability measures, in predicting mathematics performance. On the other hand, Null Hypotheses 1a), 1b), and 2a) must be accepted. Measures of concept formation, reflective activity with concepts, and test anxiety do not appear to have made a significant contribution in improving the efficiency of predicting mathematics performance scores.

The best weighted sum of DAT measures for predicting the mathematics criterion scores was $0.33X_{NA} + 0.24X_{VR}$, giving a multiple correlation of 0.6439 with the criterion. This ratio of the weights is suggestive of the VR+NA combination frequently used as a measure of scholastic aptitude and found to correlate well with mathematics achievement. The correlation between the VR+NA and SMU-II scores was 0.6405. With VR+NA and the remaining seven variables as candidates for inclusion in a regression equation to predict SMU-II scores, the "best fit" regression equation was found to involve the VR+NA and SK6(1) variables as predictors. The optimum weighting of these produced a multiple correlation of 0.6650 with the criterion, indicating that 44.23 per cent of the variance had been accounted for. Table X displays the values for the appropriate statistical tests involved in selecting the best fit prediction equation using VR+NA as one of the predictors. The

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TABLE X

SIGNIFICANT PREDICTORS OF SMU-II SCORES IN THE ORDER ENTERED DURING STEPWISE REGRESSION ANALYSIS WHEN ONE OF THE PREDICTORS WAS VR+NA

Step, n	Predictors Included in Regression Equation	Multiple Correlation, R	F_R	P_R	Percentage of Variance Accounted for	t_n
1	VR+NA	0.6405	89.73	0.000	41.02	——
2	VR+NA, SK6(1)	0.6650	50.76	0.000	44.23	2.71
3	VR+NA, SK6(1), AR	0.6707	34.61	0.000	44.98	1.32

Best Prediction Equation: $\hat{X}_{SMU\text{-}II} = {}^{-}0.752 + 0.234X_{VR+NA} + 0.195X_{SK6(1)}$

Coefficient $\underline{t}$ ratios: 7.00 2.71

$t_{.01}(127\underline{+}1df) = 2.36$

multiple correlation for this weighting (0.6650) was, for any practical purpose, not significantly different from the optimum weighting of NA, VR, and SK6(1) scores that produced a 0.6696 multiple correlation with the criterion.

The best weighted sum of Skemp's test scores was $0.21X_{SK6(1)} + 0.27X_{SK6(2)}$, producing a multiple correlation with the SMU-II scores of 0.5414. The ratio of the weights in this combination is strongly suggestive of the sum SK6(1)+SK6(2). Scores produced from the sum SK6(1)+SK6(2) had a correlation of 0.5405 with the criterion. As with the VR and NA scores, a simple sum of the SK6(1) and SK6(2) scores is very nearly as efficient as the optimum weighting.

The multiple correlation coefficient obtained from a regression equation using VR+NA and SK6(1)+SK6(2) as predictors was 0.6693, accounting for 44.80 per cent of the variance of the SMU-II scores. Table XI displays a summary of the findings from the stepwise regression analysis in which VR+NA and SK6(1)+SK6(2) were potential predictors. The resulting "best fit" prediction equation (see Table XI) suggests that a 2 to 1 weighting of the VR+NA and SK6(1)+SK6(2) scores might give a useful index of a student's ability in mathematics at the grade eight level.

TABLE XI

PREDICTION OF SMU-II SCORES USING VR+NA AND SK6(1)+SK6(2) VARIABLES

Step, n	Predictors Included in Regression Equation	Multiple Correlation, R	F_R	P_R	Percentage of Variance Accounted for	t_n
1	VR+NA	0.6405	89.73	0.000	41.02	——
2	VR+NA, SK6(1)+SK6(2)	0.6693	51.94	0.000	44.80	2.96
3	VR+NA, SK6(1)+SK6(2), AR	0.6739	35.23	0.000	45.42	1.20

Best Prediction Equation: $\hat{X}_{SMU\text{-}II} = 0.322 + 0.216X_{VR+NA} + 0.106X_{SK6(1)+SK6(2)}$

Coefficient *t* ratios: 6.02 2.96

$t_{.01}(127 \pm 1df) = 2.36$

Such a combination of scores would be more convenient for practical purposes than an optimum weighting and would very likely be as useful. The $2X_{VR+NA} + X_{SK6(1)+SK6(2)}$ combination of scores had a 0.6693 correlation with the SMU-II scores, which to the fourth decimal place is identical with the correlation from the optimum weighting of the VR+NA and SK6(1)+SK6(2) scores. A follow-up study using grade nine Province of Alberta mathematics examination scores as the criterion produced evidence confirming that the 2 to 1 weighting of the VR+NA and SK6(1)+SK6(2) scores gives a useful index of mathematics potential (See the Appendix for details of the findings from the follow-up study.).

To investigate Null Hypotheses 1e) and 2b), all possible products of pairs of the ten variables were formed and these were introduced as variables (in addition to the original variables) for possible inclusion in a regression equation for predicting the SMU-II scores. The product variable with the numerically greatest correlation with the criterion was VR×NA with a correlation of 0.6548. The product variables regression equation that produced the "best fit" and the appropriate statistical test values are displayed in Table XII. The multiple correlation coefficient from the prediction equation involving VR×NA and AR×SK6(1)

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TABLE XII

SIGNIFICANT PREDICTORS OF SMU-II SCORES IN THE ORDER ENTERED DURING STEPWISE REGRESSION ANALYSIS WHEN PRODUCTS OF PAIRS OF VARIABLES USED

Step, n	Predictors Included in Regression Equation	Multiple Correlation, R	F_R	P_R	Percentage of Variance Accounted for	t_n
1	VR×NA	0.6548	96.85	0.000	42.88	——
2	VR×NA, AR×SK6(1)	0.6857	56.77	0.000	47.01	3.16
3	VR×NA, AR×SK6(1), SK4(1)×TASC	0.6915	38.79	0.000	47.82	1.40

Best Prediction Equation: $\hat{X}_{SMU\text{-}II} = 5.945 + 0.0087X_{VR\times NA} + 0.0047X_{AR\times SK6(1)}$

Coefficient *t* ratios: 6.09 3.16

$t_{.01}(128\underline{+}1df) = 2.36$

was 0.6857, accounting for 47.01 per cent of the variance of the SMU-II scores. Since the AR×SK6(1) variable did make a significant contribution to the prediction equation, rejection of Null Hypothesis 1e) would be justified. However, Null Hypothesis 2b) must be accepted. Skemp's operations formation test scores interacting with the DAT abstract reasoning scores significantly increased the efficiency of prediction of mathematics performance from the VR×NA combination. The product (interaction) variables were better predictors than the original variables, suggesting that the effects of such interactions as a high verbal ability score combined with a low quantitative ability score are masked when simple weighted sums of the scores are used for prediction of mathematics performance. Interactions between the test anxiety scores and any of the other predictors did not appear to have any marked relationship with mathematics performance scores. Even transforming the TASC scores to the arcsines of numbers corresponding to the values of the sine function in its first half cycle did not produce an anxiety score that increased the predictive efficiency of any of the combinations of scores so far reported. (This was done with a view to approximating the inverted U-shaped plot of anxiety scores against performance scores that was reported in the literature.).

Though the product variables did produce somewhat higher correlations with the criterion than the original variables, the 0.6857 correlation obtained from the product variables was, for any practical purpose, not significantly greater than the 0.6696 correlation obtained from the optimum weighting of the VR, NA, and SK6(1) scores. There appears to have been no practical increase in predictive efficiency resulting from the use of product variables.

Stepwise regression analyses similar to those described so far were carried out using the score from each of the five SMU-II subtests (Measurement, Numeration, Number Systems, Geometry, and Problem Solving) as the criterion, but no variables other than VR, NA, SK6(1), or SK6(2) were found to be useful predictors. Prediction patterns similar to those described for the prediction of the SMU-II scores were found to emerge.

Skemp's calculations of a series of partial correlations between scores from his tests and those from a mathematics criterion produced some interesting insights into the nature of his tests in relation to the mathematics criterion. Following Skemp's method, a series of first order partial correlations (i.e., correlations between two variables with the effects of a third removed) were calculated from the data of the study presently being reported. The partial

correlations are displayed in Table XIII. For a sample consisting of 131 sets of observations, a first order partial correlation greater than or equal to 0.204 would be considered significantly different from zero (0.01 level test). Most relevant to Null Hypothesis 1 is an examination of what happens to the correlations between the SMU-II scores and the scores on each of Skemp's tests as the effects of each of the other variables are removed in turn. As can be seen from Table XIII, partialing out the portions of the correlation attributable to relationships with any one of the other predictor variables (having a significant correlation with the criterion) produces correlations between the SMU-II scores and the SK6(1) or SK6(2) scores that are still significantly different from zero at the 0.01 level. This finding tends to support the contention that the relationship between a student's performance on Skemp's test and his performance on a mathematics test can be attributed to factors other than the abstract nature common to the tests or the general scholastic aptitude of the student. The partial correlations between SK4(2) and the criterion after the effects attributable to the VR or NA scores are removed are not significantly different from zero, suggesting that general scholastic aptitude accounts for a portion of the correlation between the

TABLE XIII

PARTIAL CORRELATIONS BETWEEN SMU-II AND VARIABLE "A" WITH THE EFFECTS OF VARIABLE "B" REMOVED

VARIABLE "B"	VARIABLE "A"							
	VR	NA	AR	SR	SK4(1)	SK4(2)	SK6(1)	SK6(2)
VR	——	0.421	0.307	0.181	0.226	0.192	0.301	0.336
NA	0.369	——	0.292	0.167	0.224	0.200	0.341	0.266
AR	0.400	0.439	——	0.129	0.233	0.229	0.277	0.316
SR	0.474	0.508	0.387	——	0.299	0.312	0.380	0.398
SK4(1)	0.472	0.509	0.405	0.263	——	0.266	0.371	0.406
SK4(2)	0.434	0.479	0.373	0.226	0.210	——	0.366	0.383
SK6(1)	0.401	0.472	0.284	0.123	0.168	0.227	——	0.289
SK6(2)	0.409	0.412	0.299	0.123	0.211	0.224	0.262	——
ZERO ORDER CORRELATIONS	0.537	0.568	0.482	0.334	0.362	0.394	0.478	0.491

Critical 0.01 level $r_{ij \cdot k} = 0.204$

SK4 scores and the criterion scores. However, the partial correlations in question are so close to the critical 0.01 level values that the latter assertion must be made with some reservations. Skemp's finding that the correlations between scores on a mathematics test and the Operations Formation scores were reduced to near zero when the effects of relations with the Manipulation of Concepts or Manipulation of Operations scores were removed was not confirmed in the present investigation.

For a sample size of 131 sets of observations, a second order partial correlation (i.e., $r_{ij \cdot kl}$) would have to be greater than 0.205 to be considered significantly different from zero at the 0.01 level. The partial correlation between the SK6(2) scores and the SMU-II scores with the effects of both the SK6(1) and SK4(2) scores removed was 0.262, a second order partial correlation coefficient significantly different from zero. This finding supports Skemp's finding (see page 155), but, as has been the case throughout the study presently being reported, the magnitude of the correlation is considerably smaller than that obtained by Skemp. Of all the possible pairs of variables whose effects could be removed from the correlation between the SK6(2) test and the criterion, only the AR and NA combination resulted in a second order correlation not

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significantly different from zero, but the correlation obtained (0.199) was very close to the critical value. In any event, interpretation of second order partial correlations is at best extremely difficult. Perhaps the most that can be said is that the correlation between the SK6(2) and SMU-II scores may be largely attributed to the requirement of a certain type of reasoning for successful performance on all four of the NA, AR, SK6(2), and SMU-II tests. Whether or not this type of reasoning corresponds to the exercise of reflective intelligence is a matter for conjecture.

Information about relationships between the eight predictor variables that had significant correlations with the criterion and the five subtest scores from the SMU-II test was obtained from an examination of a series of canonical correlations between the two sets of variables. The matrix of Pearson product-moment correlations from which the canonical correlations were derived is displayed in Table XIV, and a summary of the results of the canonical correlation analysis is presented in Table XV (page 236). The summary of the analysis includes listings of the canonical correlations produced by each weighting system, the chi-square values used for testing the significance of each correlation, and the set of weights used in each of the weighting

TABLE XIV

CORRELATIONS AMONG THE MATHEMATICS PERFORMANCE PREDICTOR SCORES AND THE SMU-II SUBTEST SCORES OF 131 GRADE EIGHT STUDENTS

	SMU-II SUBTESTS				
	MEASURE-MENT	NUMBER SYSTEMS	NUMER-ATION	GEOMETRY	PROBLEM SOLVING
MEASUREMENT	1.000	0.245	0.316	0.307	0.285
NUMBER SYSTEMS	0.245	1.000	0.410	0.407	0.210
NUMERATION	0.316	0.410	1.000	0.342	0.149
GEOMETRY	0.307	0.407	0.342	1.000	0.423
PROBLEM SOLVING	0.285	0.210	0.149	0.423	1.000
VR	0.388	0.433	0.254	0.415	0.320
NA	0.458	0.293	0.339	0.483	0.324
AR	0.403	0.359	0.324	0.284	0.245
SR	0.309	0.238	0.185	0.244	0.147
SK4(1)	0.206	0.171	0.255	0.298	0.276
SK4(2)	0.252	0.280	0.168	0.371	0.256
SK6(1)	0.427	0.397	0.229	0.323	0.237
SK6(2)	0.347	0.344	0.300	0.392	0.261

TABLE XV

SUMMARY OF CANONICAL CORRELATION ANALYSIS

WEIGHTING SYSTEM	CANONICAL CORRELATION	CHI-SQUARE	df	P
1	0.702	114.47	40	<0.01
2	0.355	30.04	28	<0.01
3	0.247	13.36	18	>0.01
4	0.196	5.57	10	>0.01
5	0.075	0.69	4	>0.01

WEIGHTING SYSTEM 1

SUBTEST	WEIGHT	PREDICTOR	WEIGHT
MEASUREMENT	-0.697	VR	-0.547
NUMBER SYSTEMS	-0.458	NA	-0.682
NUMERATION	-0.109	AR	-0.156
GEOMETRY	-0.495	SR	0.029
PROBLEM SOLVING	-0.218	SK4(1)	0.022
		SK4(2)	-0.022
		SK6(1)	-0.443
		SK6(2)	-0.111

WEIGHTING SYSTEM 2

SUBTEST	WEIGHT	PREDICTOR	WEIGHT
MEASUREMENT	-0.300	VR	-0.215
NUMBER SYSTEMS	-0.670	NA	0.383
NUMERATION	0.507	AR	-0.067
GEOMETRY	0.382	SR	-0.027
PROBLEM SOLVING	0.241	SK4(1)	0.596
		SK4(2)	-0.047
		SK6(1)	-0.639
		SK6(2)	0.189

systems that produced significant canonical correlations. As can be seen from Table XV, the first two weighting systems produced significant canonical correlations between the two sets of variables whereas the remaining weighting systems did not. The weights assigned in Weighting System 1, which produced a canonical correlation of 0.702, indicate that the Measurement, Number Systems, and Geometry scores were most closely related to the VR, NA, and SK6(1) scores. It appears that one's verbal reasoning ability, numerical ability, and operations formation skill are much more closely related to his performance in dealing with measurement, geometry, and number systems problems than are his abilities in, say, perceiving spatial relations or forming and manipulating concepts, at least insofar as these abilities are measured by the test battery used in the investigation. Differences among the weights assigned to the other variables are less striking. The second weighting system suggests close relationships between Number Systems and SK6(1) scores and between Numeration and SK4(1) scores, indicating that facility with number systems questions (perhaps) requires operation formation skills to a greater extent than do the other types of questions in the SMU-II tests and that a similar statement could be made in terms of numeration questions and concept formation.

II. AGE LEVEL BY SEX CONTRASTS IN TERMS OF CONCEPT AND OPERATIONS FORMATION AND MANIPULATION

Contrasts were made among the mean scores obtained by boys and girls aged ten through sixteen on each of Skemp's four tests in order to test Null Hypothesis 3, which is here re-stated for reference.

> Null Hypothesis 3. For school children in grades five through eleven, there are no significant differences among the group mean scores obtained by students in age categories from ten to sixteen, or between boys' and girls' group mean scores, on tests of
> a) concept formation,
> b) reflective activity with concepts,
> c) operations formation, or
> d) reflective activity with operations,
> and there are no significant interaction effects among the age level and sex categories.

Concept Formation (SK4(1))

The SK4(1) test cell and group means are displayed in the appropriate age and sex categories in Table XVI, and a summary of the two-way analysis of variance carried out on the SK4(1) scores is presented in Table XVII. The column headings SS, df, MS, and F in Table XVII stand for "sums of squares," "degrees of freedom," "mean squares," and "*F* ratios," respectively.

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TABLE XVI

SK4(1) TEST CELL AND GROUP MEANS

SEX	AGE GROUP							GROUP MEANS
	10	11	12	13	14	15	16	
BOY	10.45	9.90	9.53	11.20	10.50	11.90	12.42	10.84
GIRL	7.89	9.80	11.30	10.84	11.33	13.36	13.16	11.10
GROUP MEANS	9.17	9.85	10.41	11.02	10.92	12.63	12.79	10.97

TABLE XVII

SUMMARY OF ANALYSIS OF VARIANCE OF SK4(1) TEST SCORES

SOURCE	SS	df	MS	F
AGE	475.38	6	79.23	6.59
SEX	5.01	1	5.01	0.42
INTERACTION	139.50	6	23.25	1.93
WITHIN	3,918.83	326	12.02	

$F_{.01}(6,326) = 2.80$ $F_{.01}(1,326) = 6.63$

Since the observed *F* ratio (6.59) for comparisons among the age level mean scores exceeded the critical 0.01 level value (2.80), the null hypothesis that there were no significant differences among the age level mean scores was rejected. The hypotheses that there was no significant difference between sex group means and that there was no significant interaction effect were accepted as the observed *F* ratios did not exceed the appropriate critical values. Accordingly, it can be asserted that there were significant differences among the age level mean SK4(1) scores, but there was no significant difference between the mean SK4(1) score obtained by the boys and that obtained by the girls. The interaction effect was not statistically significant.

Since significant differences were found among the seven age level mean scores on the SK4(1) test, the group means were compared two at a time to discover which group means were significantly different from one another. By substituting $t_{.01}(117df) = 2.36$ for t in the *t* ratio formula given on page 213, a 0.01 level critical mean difference of 1.75 for the SK4(1) scores was obtained. Below, the age level group means are listed in rank order with the means that were not significantly different from one another underlined by a common line.

	AGE LEVEL						
	10	11	12	14	13	15	16
SK4(1) GROUP MEAN	9.17	9.85	10.41	10.92	11.02	12.63	12.79

Figure 2 displays a plot of the mean scores on the SK4(1) test against age level. Considering that the mean score obtained by the fourteen-year-olds was not significantly different from those of the eleven to fifteen-year-olds, there is some justification for asserting that there was a steady increase in the level of performance on the SK4(1) test with increasing age of the children tested. The fifteen to sixteen-year-olds performed significantly better on the concept formation test than the ten to twelve-year-olds.

<u>Reflective Action with Concepts (SK4(2))</u>

The SK4(2) cell and group means are listed in Table XVIII in the appropriate age and sex categories. A summary of the analysis of variance carried out on the SK4(2) scores is presented in Table XIX.

As the observed *F* ratio (10.05) for comparisons among the age level mean scores exceeded the critical value, the hypothesis that there were no significant differences among

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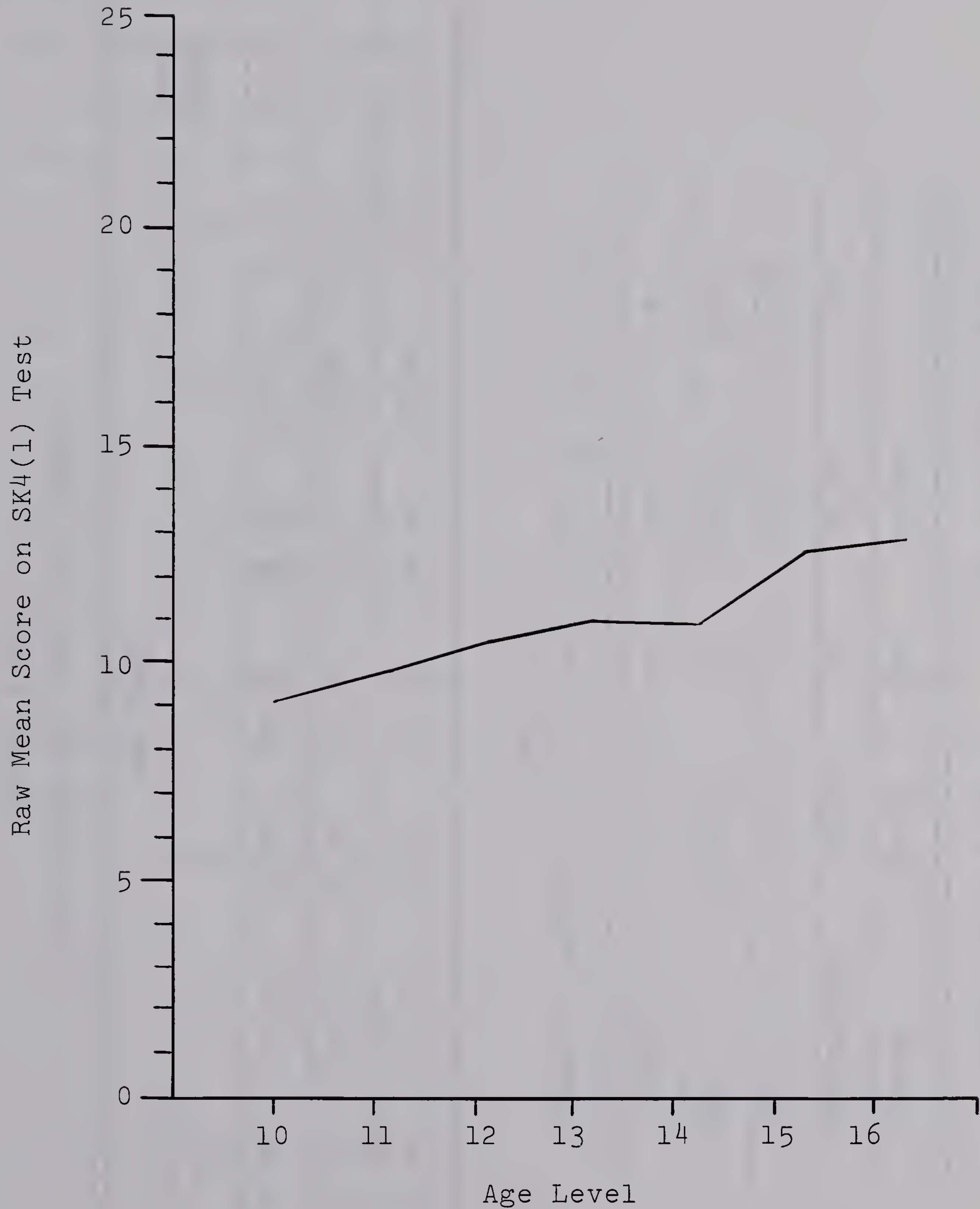

FIGURE 2

THE RELATIONSHIP BETWEEN MEAN SK4(1) SCORE AND AGE LEVEL OF STUDENTS TESTED

TABLE XVIII

SK4(2) TEST CELL AND GROUP MEANS

SEX	AGE GROUP							GROUP MEANS
	10	11	12	13	14	15	16	
BOY	12.91	13.24	13.44	16.32	13.36	17.70	19.69	15.24
GIRL	11.16	14.27	16.80	17.04	15.33	18.86	19.52	16.14
GROUP MEANS	12.03	13.75	15.12	16.68	14.35	18.28	19.61	15.69

TABLE XIX

SUMMARY OF ANALYSIS OF VARIANCE OF SK4(2) TEST SCORES

SOURCE	SS	df	MS	F
AGE	1,850.40	6	308.40	10.05
SEX	62.11	1	62.11	2.03
INTERACTION	169.66	6	28.28	0.92
WITHIN	10,000.31	326	30.68	

$F_{.01}(6,326) = 2.80$ $F_{.01}(1,326) = 6.63$

the age level mean scores was rejected. The hypotheses that there was no significant difference between the sex group means and that there was no significant interaction effect were accepted because the observed F ratios did not exceed the appropriate critical values. Accordingly, there were significant differences among the age level mean SK4(2) scores, but there was no significant difference between the mean SK4(2) score obtained by the boys and that obtained by the girls. The interaction effect was not statistically significant.

Since significant differences were found among the age level mean scores on the SK4(2) test, the group means were compared two at a time to discover which means were significantly different from one another. The difference between two of the SK4(2) means would have had to exceed 2.79 before the means could be considered significantly different at the 0.01 level (following the procedure described on page 240). The age level group means on the SK4(2) test are recorded below in rank order with the means that were not significantly different from one another underlined by a common line.

	AGE LEVEL						
	10	11	14	12	13	15	16
SK4(2) GROUP MEAN	12.03	13.75	14.35	15.12	16.68	18.28	19.61

Figure 3 displays a plot of the mean scores on the SK4(2) test against age level. As the mean score obtained by the fourteen-year-olds was not significantly different from those of the twelve and thirteen-year-olds, one could say that the SK4(2) mean scores increased steadily with increasing age of the subjects tested. The fifteen and sixteen-year-olds performed at a significantly higher level on the reflective action with concepts test than the ten to twelve-year-olds. Considering that a similar pattern was found for the SK4(1) scores, it is difficult to interpret what the meaning of the steady increase in SK4(2) scores is. Do the older students score higher on the SK4(2) test because they are more capable of reflective thought than the younger students, or do the older students score higher because they have a better grasp of the concepts learned in SK4(1) and required in SK4(2)? Perhaps both factors are involved. In an attempt to shed some light on this question, the SK4(1) means were arbitrarily rescaled so that the grand mean for the sample would be 15 and the standard deviation, 5. The SK4(2) scores were arbitrarily rescaled to have a mean of 17 and a standard deviation of 5, (so that the corresponding plotted mean scores would be close and yet not overlap). The rescaled means for the two tests are plotted against age level in Figure 4. As can be seen from the Figure, the

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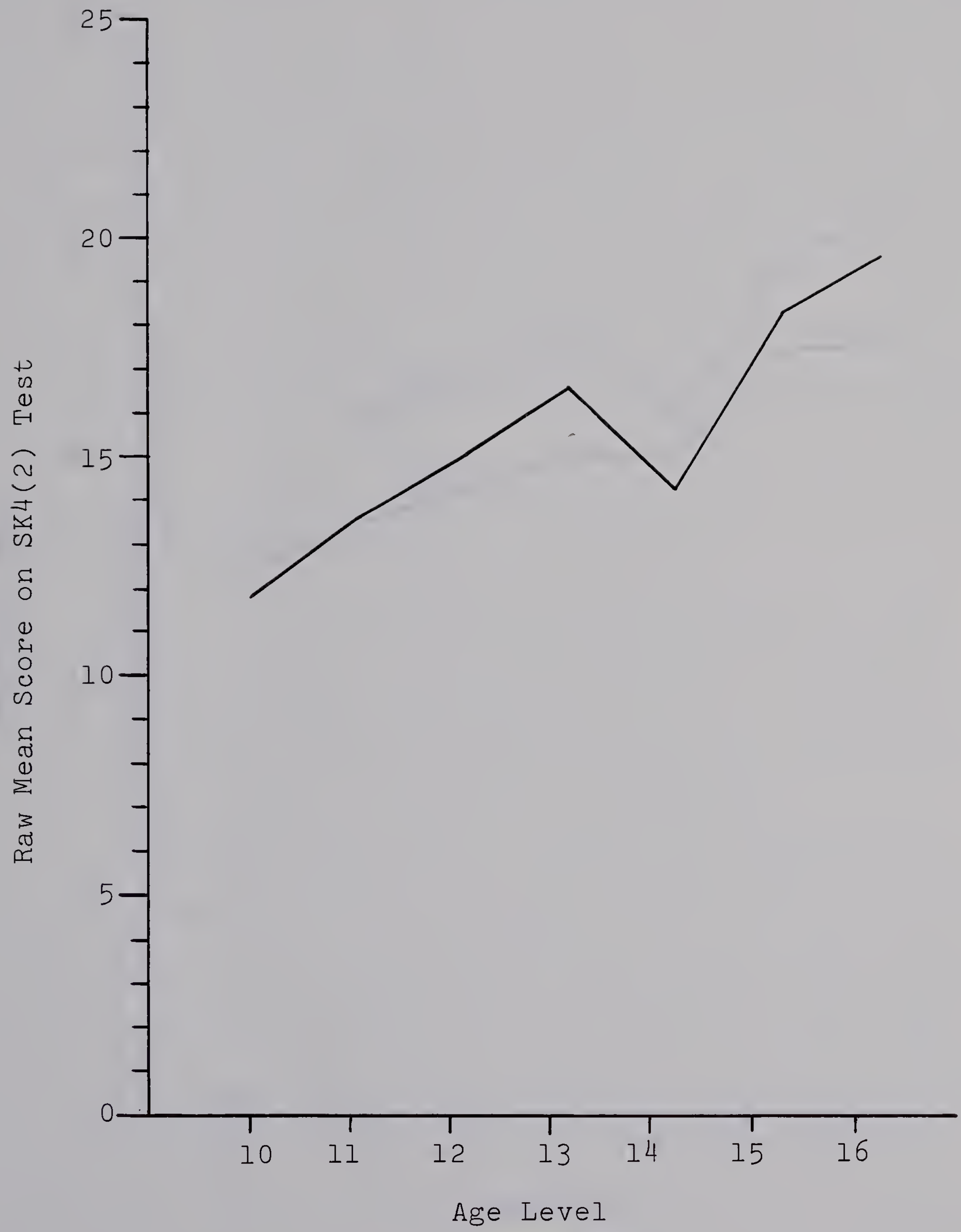

FIGURE 3

THE RELATIONSHIP BETWEEN MEAN SK4(2) SCORE AND AGE LEVEL OF STUDENTS TESTED

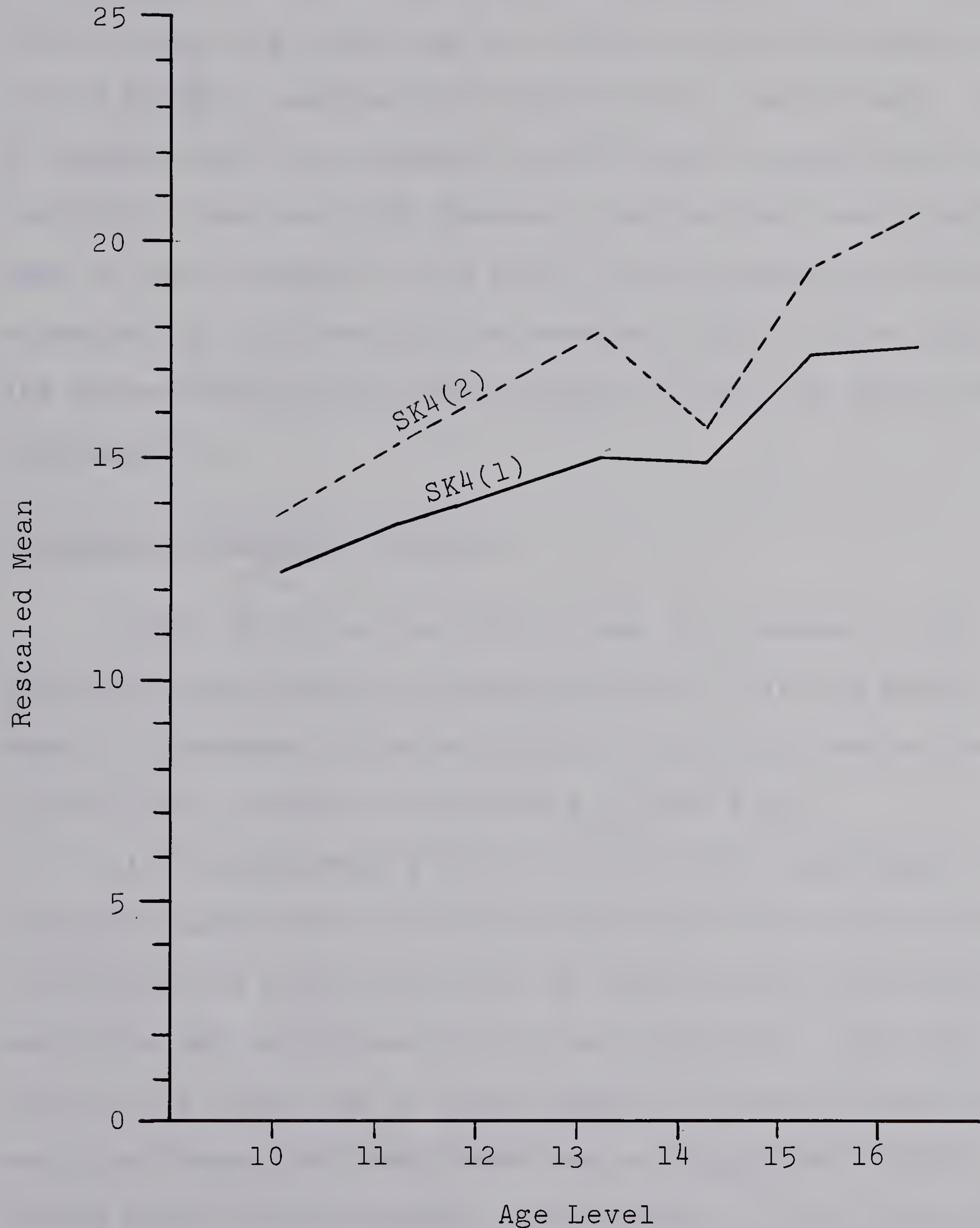

FIGURE 4

COMPARISON OF SK4(1) AND SK4(2) MEAN GROWTH PATTERNS

SK4(2) means increased at a slightly faster rate than the SK4(1) means did (ignoring the fourteen-year-old depression). Only a grossly qualitative interpretation can be made, but it appears that the younger children may do more poorly on the SK4(2) test not only because they may not have grasped some of the concepts on the SK4(1) test as well as the older students did but perhaps also because they could not operate reflectively with these concepts as well as their older counterparts.

Operations Formation (SK6(1))

Table XX lists the SK6(1) test cell means in the appropriate age and sex categories as well as the group means. A summary of the analysis of variance carried out on the SK6(1) scores is presented in Table XXI.

As the observed $\underline{F}$ ratio (13.89) for comparisons among the age level mean scores exceeded the critical value, the hypothesis that there were no significant differences among the age level mean scores was rejected. The hypotheses that there was no significant difference between the sex group means and that there was no significant interaction effect were accepted. Accordingly, it was found that there were significant differences among the age level mean SK6(1) scores, but there was no significant difference

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TABLE XX

SK6(1) TEST CELL AND GROUP MEANS

SEX	AGE GROUP							GROUP MEANS
	10	11	12	13	14	15	16	
BOY	16.36	17.86	19.74	21.28	18.55	24.95	23.81	20.36
GIRL	16.53	17.03	17.95	21.44	20.54	23.14	24.68	20.19
GROUP MEANS	16.44	17.45	18.84	21.36	19.54	24.05	24.24	20.27

TABLE XXI

SUMMARY OF ANALYSIS OF VARIANCE OF SK6(1) TEST SCORES

SOURCE	SS	df	MS	F
AGE	2,469.26	6	411.54	13.89
SEX	2.37	1	2.37	0.08
INTERACTION	128.32	6	21.39	0.72
WITHIN	9,656.96	326	29.62	

$F_{.01}(6,326) = 2.80$ $F_{.01}(1,326) = 6.63$

between the mean SK6(1) score obtained by the boys and that of the girls. The interaction effect was not statistically significant.

As significant differences were found among the age level mean scores on the SK6(1) test, the group mean scores were contrasted two at a time. The critical 0.01 level mean difference for the SK6(1) scores was 2.74. The age level group means from the SK6(1) test are listed below in rank order with the means that were not significantly different from one another underlined by a common line.

AGE LEVEL

	10	11	12	14	13	15	16
SK6(1) GROUP MEAN	16.44	17.45	18.84	19.54	21.36	24.05	24.24

Figure 5 displays a plot of the mean scores on the SK6(1) test against age level. The pattern was similar to those found for the SK4(1) and SK4(2) scores. Again, it seems reasonable to describe the trend as a steady increase in SK6(1) mean score levels with increasing age of the subjects tested. The fifteen and sixteen-year-olds performed at a significantly higher level than the ten to twelve-year-olds on the SK6(1) test.

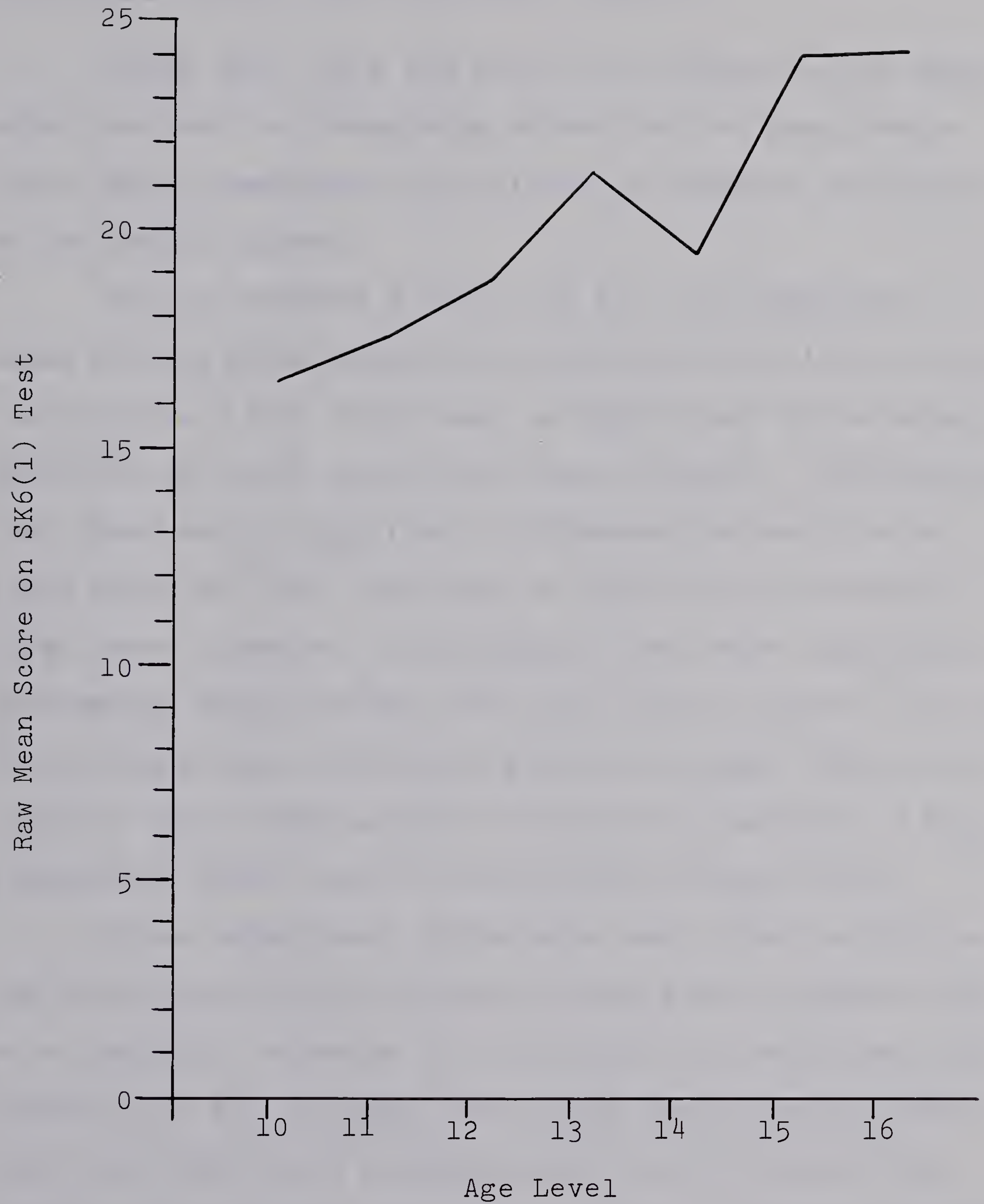

FIGURE 5

THE RELATIONSHIP BETWEEN MEAN SK6(1) SCORE
AND AGE LEVEL OF STUDENTS TESTED

<u>Reflective Action with Operations (SK6(2))</u>

Table XXII lists the SK6(2) cell means in the appropriate age and sex categories as well as the group means. Table XXIII summarizes the analysis of variance carried out on the SK6(2) scores.

As the observed <u>F</u> ratio (12.33) for comparisons among the age level mean scores exceeded the critical value, the hypothesis that there were no significant differences among the age level mean scores was rejected. The hypotheses that there was no significant difference between the sex group means and that there was no significant interaction effect were accepted. Accordingly, there were significant differences among the age level mean SK6(2) scores, but there was no significant difference between the mean SK6(2) score obtained by the boys and that obtained by the girls. The interaction effect was not statistically significant.

Since significant differences were found among the age level mean scores, the group means were contrasted two at a time with reference to a critical 0.01 level mean difference of 4.20. The age level group means from the SK6(2) test are listed below in rank order with the means that were not significantly different from one another underlined by a common line.

TABLE XXII

SK6(2) TEST CELL AND GROUP MEANS

SEX	AGE GROUP							GROUP MEANS
	10	11	12	13	14	15	16	
BOY	13.55	13.21	14.85	18.88	13.73	20.95	24.11	17.04
GIRL	9.58	11.43	14.90	17.04	16.21	20.57	23.06	16.11
GROUP MEANS	11.56	12.32	14.88	17.96	14.97	20.76	23.59	16.58

TABLE XXIII

SUMMARY OF ANALYSIS OF VARIANCE OF SK6(2) TEST SCORES

SOURCE	SS	df	MS	F
AGE	5,137.86	6	856.31	12.33
SEX	65.62	1	65.62	0.95
INTERACTION	259.12	6	43.19	0.62
WITHIN	22,643.28	326	69.46	

$F_{.01}(6,326) = 2.80$ $F_{.01}(1,326) = 6.63$

AGE LEVEL

	10	11	12	14	13	15	16
SK6(2) GROUP MEANS	11.56	12.32	14.88	14.97	17.96	20.76	23.59

Figure 6 displays a plot of the SK6(2) mean scores against age level. The pattern is similar to those discussed for the previous three tests, and it can be stated that, in general, the SK6(2) mean scores increase steadily with the age level of the subjects tested. No sudden, significant jumps from one mean score to the next are in evidence. It is noted that the fifteen and sixteen-year-olds performed at a significantly higher level on the reflective action with operations test than the ten to twelve-year-olds did. As with the SK4 test scores, the growth pattern seems similar for both the SK6(1) and SK6(2) mean scores, making an adequate interpretation of the increasing SK6(2) scores difficult. To help contrast the patterns, the mean scores on the two parts of the SK6 test were arbitrarily rescaled so that the sample standard deviation for both would be 5 and the SK6(1) mean would be 15 while the SK6(2) mean would be 17. The rescaled means are plotted against age level in Figure 7. There does not appear to have been a divergence between the means with increasing age of the subjects tested, making one suspect that the younger children do more poorly

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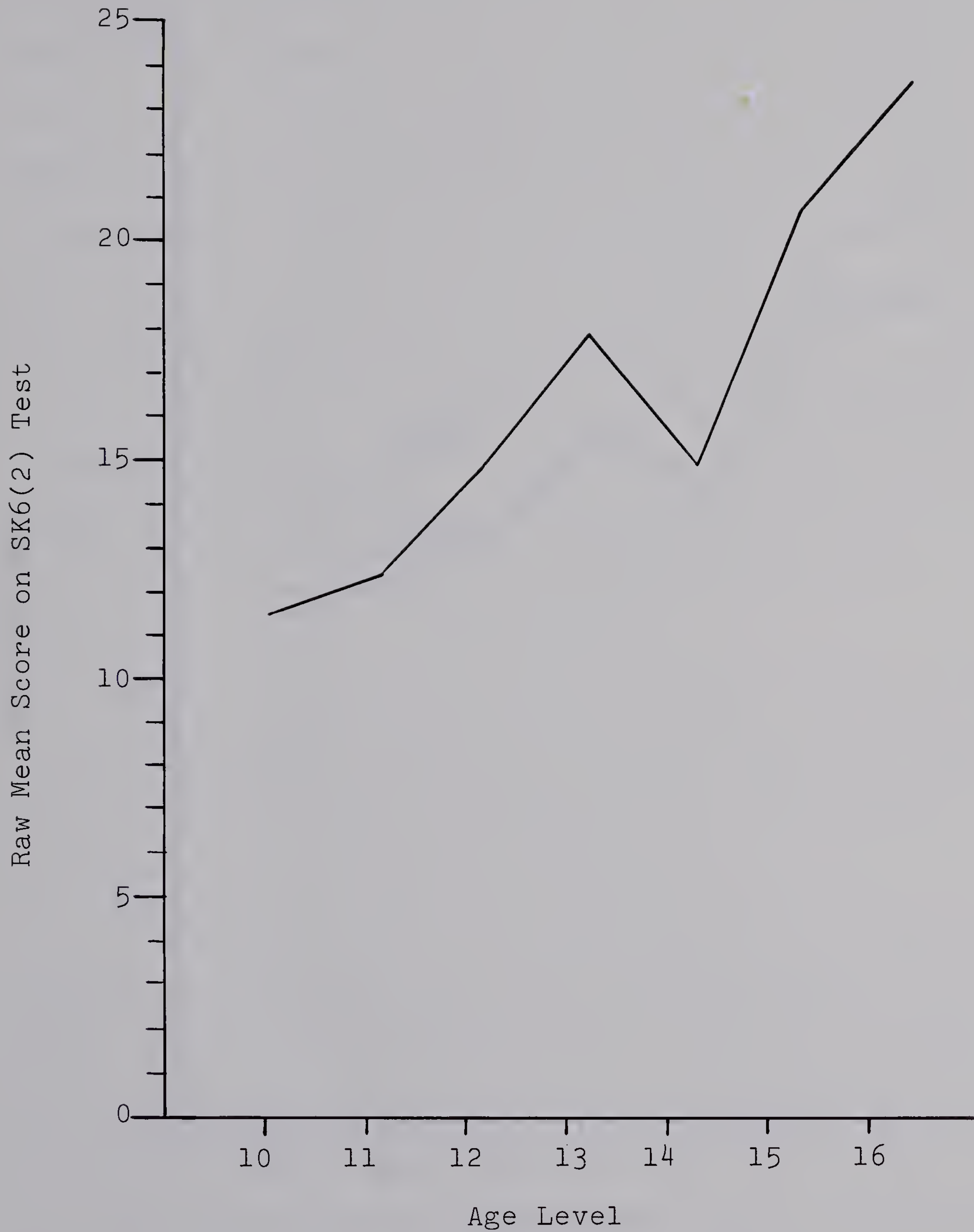

FIGURE 6

THE RELATIONSHIP BETWEEN MEAN SK6(2) SCORE
AND AGE LEVEL OF STUDENTS TESTED

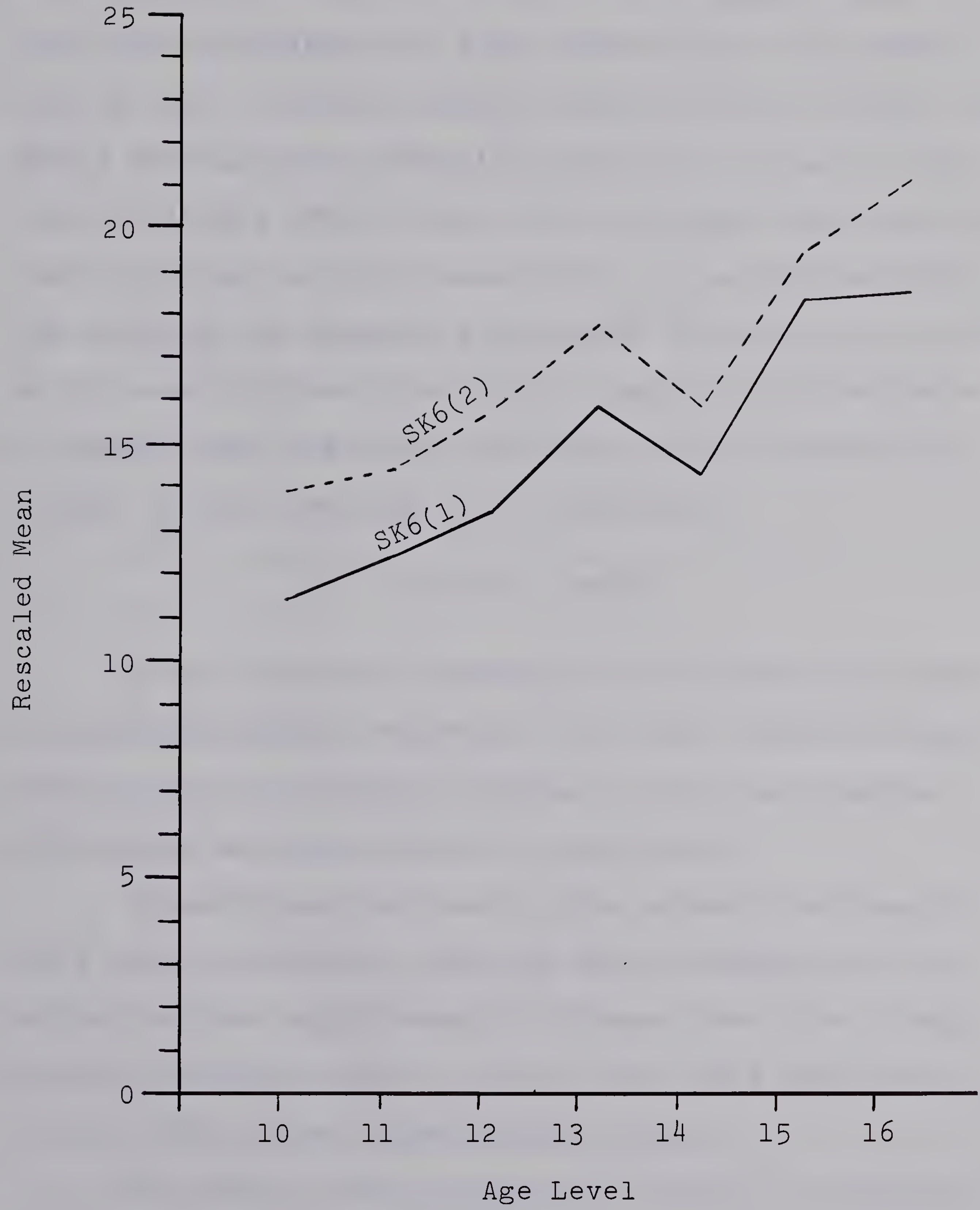

FIGURE 7

COMPARISON OF SK6(1) AND SK6(2) MEAN
GROWTH PATTERNS

on the SK6(2) test than the older ones perhaps because they have not grasped the basic operations in the SK6(1) test as well. Another possible interpretation is that the SK6(1) test involves reflective activity similar to that involved in the SK6(2) test, and the younger children are less able than the older ones to do the reflective thinking required for adequate performance on each of the tests. As has been discussed previously, there is some evidence to suggest that the SK6(1) test does require reflective thought in the answering of its questions.

III. CHAPTER SUMMARY

In all cases in the analysis of the data collected in the investigation reported, 0.01 level critical values were employed in assessing whether or not the observed differences were statistically significant.

It was found that each of the scores from Skemp's tests had a correlation with the SMU-II mathematics criterion that was significantly different from zero. This was also true for each of the four DAT tests used but not for the TASC anxiety questionnaire scores.

The results from a number of "stepwise regression" analyses demonstrated that the SK6(1) and SK6(2) scores made significant contributions to the efficiency of

predicting mathematics performance scores over and above the contributions made by the best predictors from the DAT battery (viz., the NA and VR scores). Skemp's SK6(1) and SK6(2) tests give measures of one's ability in operations formation and reflective activity with operations, respectively. SK4(1) and SK4(2), Skemp's tests of concept formation and reflective activity with concepts, were not found to make significant contributions to the mathematics score prediction equations. The AR and SR tests from the DAT battery and TASC anxiety questionnaire were also not found to be useful predictors of mathematics performance at the grade eight level.

A particularly interesting finding was that a score arrived at by weighting the sum of the VR and NA scores and the sum of the SK6(1) and SK6(2) scores in the ratio 2 to 1 gave a useful index of mathematics potential.

The use of products of pairs of the predictor variables available in the investigation was found not to produce a practical increase in mathematics score prediction efficiency.

Skemp's findings with regard to partial correlations between his tests and a mathematics criterion were generally confirmed. An exception was that the correlation between the operations formation score and the mathematics criterion

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was not found to be reduced to near zero in the present study when the effects of the reflective action with concepts or the reflective action with operations scores were partialed out.

An examination of the relationships among the various predictor variables and the five subtest scores from the SMU-II test by means of canonical correlations demonstrated close connections among the VR, NA, SK6(1), Measurement, Geometry, and Number Systems scores. Similarly, the SK4(1) and SK6(1) scores were found to be closely connected with the Numeration and Number Systems scores, respectively.

Two-way analyses of variance carried out on the SK4(1), SK4(2), SK6(1), and SK6(2) scores categorized according to age (ten to sixteen) and sex (boy or girl) demonstrated that the fifteen and sixteen-year-olds scored significantly higher than the ten to twelve-year-olds on each of Skemp's tests. The thirteen and fourteen-year-olds appeared to have mean scores that were intermediate to the mean scores of the younger and older groups. The differences between the mean scores obtained by the boys and those obtained by the girls on each of the four tests were not statistically significant. The growth patterns of the mean scores in terms of increasing age of the subjects tested indicated that not only did the younger children have

a poorer grasp of the basic concepts and operations involved in Skemp's tests, but they were less able to operate reflectively with these concepts and operations.

CHAPTER V

SUMMARY, CONCLUSIONS, LIMITATIONS, IMPLICATIONS FOR MATHEMATICS EDUCATION, AND IMPLICATIONS FOR FURTHER RESEARCH

I. PURPOSE OF THE INVESTIGATION

The purpose of the investigation was to gather theoretical and experimental evidence to provide a basis for assessing R. R. Skemp's theory of mathematics learning. The theoretical evaluation of Skemp's position took the form of a review of the literature in which his theory was examined in the light of other contemporary learning theories. The empirical part of the investigation was so designed that data could be gathered and analyzed to shed light on the following questions:

1. Is the presence of reflective intelligence a factor of enough importance for success in mathematics that the addition of measures of reflective intelligence to measures of general intelligence significantly improves the prediction of one's performance in mathematics?

2. What effects on the relationships between measured reflective intelligence and mathematics performance result from taking into account student anxiety toward testing situations?

3. Are there any significant differences among the mean levels of reflective intelligence exhibited by boys or girls in age categories from ten to sixteen?

II. THE FINDINGS FROM THE REVIEW OF THE LITERATURE

The review of the literature pointed out inadequacies of the classical learning theories in explaining mathematics learning, described Piaget's theory of cognitive growth, indicated how the theories of Bruner, Dienes, and Skemp were derived from a Piagetian point of view, and marshalled theoretical and empirical support for Skemp's theory of mathematics learning.

The failure of classical learning theories to account adequately for purposive behaviour and for the effects of previous learning on present learning was noted. Though learning theories in the behaviouristic frame of reference could be related to rote learning through reward or punishment, such learning was viewed as of minor concern in classroom situations.

Piaget's theory, which employs such constructs as schemata, assimilation, accommodation, operations, reversibility, and equilibration, was described as being more adequate in accounting for the effects of past learning and purposive behaviour. Implications for mathematics education

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were found in Piaget's descriptions of cognitive development and the factors influencing such development.

The notion of Bruner and Dienes regarding recurring learning cycles which carry the learner to higher levels of abstraction and generalization was viewed as a useful interpretation and extension of Piaget's theory of cognitive growth. Bruner's descriptions of concept development and Dienes' concrete embodiments of mathematical concepts and operations were considered fruitful in terms of classroom application.

Skemp's three-part theory of mathematics learning was described in detail and support for its main ideas was found in the theories of Piaget, Bruner, and Dienes. Stated briefly, the main features of Skemp's theory are:

1. A description of mathematics learning as an activity requiring the exercise of reflective intelligence,

2. A description of an efficient method for building mathematical concepts, and

3. A description of schematic learning as essential to continued success and progress in mathematics learning.

Skemp's treatment of the nature of reflective intelligence, concept development, and schematic learning has been found to parallel similar notions in other theories derived from a Piagetian point of view. Providing students with experiences conducive to the building of operational thinking

structures and ensuring that explanations are given in terms that can be assimilated by the student have been cited as two useful recommendations growing out of Skemp's theory. In addition, empirical support for some of Skemp's ideas has been found in a review of research studies in mathematics education.

III. THE NATURE OF AND FINDINGS FROM THE EXPERIMENTAL INVESTIGATION

Six classes of grade eight students participated in the main part of the investigation on which the present report is based. In May and June, 1966, the students were administered a test battery consisting of the *Verbal Reasoning*, *Numerical Ability*, *Abstract Reasoning*, and *Space Relations* tests from the *Differential Aptitude Tests* (DAT) battery; Sarason's *Test Anxiety Scale for Children*; Skemp's *Concept Formation*, *Reflective Action with Concepts*, *Operations Formation*, and *Reflective Action with Operations* tests; and a *Special Mathematical Understandings* test.

Throughout the statistical analyses of the data 0.01 level critical values were used. Stepwise regression analyses of the 131 complete sets of test scores obtained from the grade eight students showed that the *Operations Formation* and *Reflective Action with Operations* scores made

significant contributions to the efficiency of predicting mathematics performance scores over and above the contributions made by the best predictors from the DAT battery (viz., Verbal Reasoning and Numerical Ability). Skemp's Concept Formation and Reflective Action with Concepts tests, the Abstract Reasoning and Space Relations tests from the DAT battery, and Sarason's Test Anxiety Scale for Children were not found to be useful predictors of mathematics performance at the grade eight level (over and above the Verbal Reasoning, Numerical Ability, Operations Formation, and Reflective Action with Operations tests). Twice the sum of the Verbal Reasoning and Numerical Ability scores plus the sum of the Operations Formation and Reflective Action with Operations scores was found to give a good index of a student's ability to achieve in school mathematics.

Calculations of partial correlations between scores on Skemp's tests and the mathematics criterion scores largely confirmed Skemp's contention that the kinds of reasoning measured by his tests and held in common with a mathematics criterion are distinct from the kinds of reasoning measured by the usual ability tests and are something other than an ability to work with abstract material. A canonical correlation analysis suggested that success in solving measurement, geometry, and number systems problems

is related to one's verbal, numerical, and operations formation abilities. Success with numeration and number systems problems appeared to be related to concept and operations formation abilities.

A second phase of the experimental investigation involved two classes of students at each of the grade five, six, seven, eight, nine, ten, and eleven levels. These students wrote Skemp's *Concept Formation*, *Reflective Activity with Concepts*, *Operations Formation*, and *Reflective Activity with Operations* tests. Two-way analyses of variance using each of the four sets of test scores, which were categorized according to age of student (ten to sixteen) and sex (boy or girl), showed that the fifteen to sixteen-year-olds scored significantly higher than the ten to twelve-year-olds on each of Skemp's tests. There were no significant differences between the boys' and girls' mean scores on the four tests.

IV. CONCLUSIONS

The theoretical assessment of Skemp's theory, as pursued in the review of the literature presented in Chapter II, has demonstrated that not only is the theory persuasive and suggestive of fruitful approaches to mathematics learning and teaching, but its main ideas are

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supported by the mainstream of contemporary thought in cognitive psychology, i.e., the Piagetian point of view. Skemp's theory has been presented and described as an insightful exposition, extension, and application to secondary school mathematics learning situations of Piagetian ideas. Skemp's discussions of the necessity of taking into account the hierarchical nature of concept formation in mathematics, of the importance of communicating with students in full awareness of the levels of thinking they have reached, and of the desirability of helping students to _formulate_ their concepts and operations have been shown to be supported by the views of Bruner and Dienes. His characterization of schematic learning as a key factor in the facilitation of present learning by relating new events to previous experiences and his emphasis on the importance of building of such cognitive structures to aid in the perception of patterns and in making sense out of new experiences was identified as being distinctly Piagetian. Skemp's notion of reflective intelligence has been found to be very useful in describing the nature of mathematics learning. His description of reflective intelligence as the ability to re-arrange, modify, and choose from among one's mental representations of previous experiences is suggestive of the type of thinking required for flexible,

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creative approaches to new problems. The concept of reflective intelligence seems to arise naturally from Piaget's writings and to fit Bruner's and Dienes' interpretations very well. It is apparent that the theoretical framework for Skemp's ideas is well founded.

On the basis of the statistical analyses carried out on the data collected in the investigation described in the present report, the answers to the questions posed on pages 261 and 262 can be framed as follows:

1. The exercise of reflective intelligence, as measured by Skemp's tests of Operations Formation and Reflective Action with Operations, was found to be of sufficient importance for success in mathematics that reflective intelligence scores did make a significant contribution to the prediction of mathematics criterion scores when added to prediction equations derived from general intelligence scores.

2. Student anxiety toward testing situations, as measured by a selection of items from Sarason's Test Anxiety Scale for Children, was not found to make a statistically significant contribution in clarifying the relationships between reflective intelligence and mathematics performance.

3. Fifteen to sixteen-year-olds were found to operate at significantly higher levels on Skemp's reflective intelligence tests than ten to twelve-year-olds.

Skemp's finding that reflective activity with operations is more closely related to success in mathematics than reflective activity with concepts was confirmed. Considering

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that concepts in mathematics are of little use unless they can be assimilated into an operational framework for solving mathematical problems, the result is understandable. Perhaps it is the static nature of the concepts used in Skemp's tests of concept formation and manipulation that causes these particular measures to play a minor role in the prediction of mathematics performance. It appears that Skemp's revision of the operations formation part of his operations test to incorporate more in the way of reflective activity has had the desired effects. It is interesting to note that of the tests administered in the investigation, next to the Numerical Ability score from the DAT battery, the best score from a forty minute testing session for predicting mathematics performance was that obtained by adding the operations formation and reflective action with operations scores. The non-arithmetical, non-verbal nature of Skemp's operations test makes it even more apparent that it is the nature of the reasoning process required in the operations test that accounts for the association between scores on that test and the mathematics criterion scores. In general, one can reasonably maintain that the importance of reflective activity in mathematical situations has been substantiated by the findings from the grade eight portion of the study and from the follow-up study using grade nine mathematics scores as

the criterion (see Appendix). The evidence gathered, both theoretical and empirical, supports acceptance of reflective intelligence as a useful construct in the field of mathematics education. It would appear that much can be gained from teaching mathematics so as to encourage students' reflective awareness and formulation of the basic mathematical concepts and operations that they have been led to abstract from their own experiences. Such teaching should facilitate progression to higher and higher levels of abstraction and generalization by aiding assimilation of higher order concepts and operations to existing schemata and accommodation of existing schemata to cope with more general applications.

The test anxiety questionnaire was included in the study in consideration of Skemp's finding that the exercise of reflective intelligence may be blocked by extreme anxiety. It was thought that, if the levels of anxiety in the testing situation were producing interference, introducing an anxiety score into the prediction equation would help to compensate for these effects. However, it was found that the anxiety scores were not significantly correlated with the mathematics performance scores, indicating that test anxiety did not appear to be operating as a significant factor influencing student levels of performance. Some rather

tenuous assumptions were made in regards to the question of anxiety in the investigation, and these are discussed in the next section, headed "Limitations."

The finding that fifteen and sixteen-year-olds performed at significantly higher levels on each of Skemp's tests than the ten to twelve-year-olds can be interpreted as generally supporting Piaget's and Skemp's descriptions of adolescent thinking as qualitatively different from that of younger children. It appears that students well into adolescence are able to operate reflectively much more satisfactorily than their younger counterparts. However, this interpretation must be somewhat tempered by the observation that the younger children had a poorer grasp than the older ones of the concepts and operations with which they were expected to act reflectively. Considering that the questions in Skemp's tests have no apparent relation to school experiences except in terms of general reasoning ability and that the kinds of tasks set involve ideas apparently well within the grasp of children of widely different ages, such tests might prove very useful as indices of reflective intelligence and mathematics potential from early primary grades up to post secondary levels.

V. LIMITATIONS

There are a number of limitations that should be kept in mind in interpreting the findings from this investigation. First of all, the assessments of Skemp's tests must be interpreted relative to the selection of the other tests included in the study. The mathematics criterion, while carefully constructed, may or may not reflect the kind of mathematics thinking that the reader believes is important at the grade eight level. (A copy of the _Special Mathematical Understandings_ test is included in the Appendix so that it may be examined.). That the grade eight mathematics criterion scores had a moderately high correlation with provincial grade nine mathematics examination scores speaks favourably for the criterion used. Although the _Differential Aptitude Tests_ battery is highly rated and has been cited as a useful predictor of school mathematics performance, other measures of general intelligence might have produced different results. Considering that it was not feasible to select the sample by random sampling of individuals rather than of classes, that the stepwise regression procedure is somewhat restrictive in terms of statistical generalizations, and that similar restrictions apply to comparisons between pairs of correlations, it is difficult to make firm statistical generalizations from the results of

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the experiment to any theoretical population. The patterns that have been found are likely to be representative of what would have been found in any random sampling of, say, grade eight classes from the Edmonton Public School population in the 1965-66 school term. Generalizations any more wide-ranging than the preceding should perhaps not be made, but for discussion purposes more general hypotheses framed in the spirit of the findings should be safer than completely unsubstantiated speculation.

Regarding the role of anxiety in influencing mathematics performance, the findings from the present study should be interpreted cautiously. Although Sarason's <u>Test Anxiety Scale for Children</u> and the original <u>Test Anxiety Scale</u> have often been shown to give useful measures of test anxiety (but not invariably), an injudicious selection of items might have been made for use in the present study (A selection had to be made because of time limitations.). The testing situations in the present study were very likely not as productive of anxiety as regular school tests, which evaluate the student for promotion purposes. However, the questionnaire items <u>were</u> phrased so that the student would respond about how he <u>usually</u> reacts to test-like situations and not specifically how he reacted to the ones immediately preceding the administration of the questionnaire. The point that

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must be made, however, is that it is quite possible that anxiety does not usually play an insignificant role in mathematics testing situations.

As far as the assessment of levels of reflective intelligence at various ages is concerned, a truly longitudinal study would of course be preferable, but somewhat impractical for an individual investigator with limited time and resources. In spite of the many confounding factors that might possibly have influenced the results, the findings are perhaps useful for guiding the framing of hypotheses about the nature of children's thought at various age levels.

VI. IMPLICATIONS FOR MATHEMATICS EDUCATION

In the interests of conciseness and since detailed discussions in support of the recommendations given below can be found in Chapter II, implications for mathematics education arising out of the present theoretical and empirical investigation are here offered in point form.

1. In view of the importance of reflective thinking for success in mathematics, every school child should be given every opportunity, from an early age, to become aware of and to gain flexibility in rearranging, restructuring, and modifying his available concepts and operations. Such activity should help lay the foundation for continued

learning with "understanding" and should make truly creative approaches to problems more likely than if rote learning of routine procedures were relied upon.

2. Mathematics instruction should proceed so that an unbroken hierarchy of basic concepts and operations is built and so that symbolization and formalization are deferred until a genuine intuitive feeling for basic ideas has been cultivated.

3. The importance of realizing what modes of thought and levels of abstraction are available to one's students cannot be overemphasized. Perhaps implementing Piaget's suggestion that teachers occasionally should set themselves a project of trying to explain mathematical ideas to an individual student would help such teachers to realize at what level the ideas are really being received. Perhaps classroom procedures could then be modified so that comprehension on the part of students would be more likely to occur.

4. Students should be encouraged to approach each new learning situation with a view to investigating whether the new experiences are consistent with existing ideas. Where conflicts arise, conscious modification of existing thought patterns should ideally be made by the student on the basis of his own experience. New notions should not

be accepted merely on the basis of verbal authority. Intelligent correction of errors involves awareness of *wherein* one's approach has been inadequate in addition to *that* it has been.

5. Great care must be taken to ensure that mathematics learning experiences are such that errors will be interpreted cognitively rather than affectively.

6. Affective conditioning of student responses should be avoided wherever possible so that flexible, multi-response, cognitively oriented approaches can be encouraged in preference to rigid, single-response, emotionally-toned thought patterns.

7. To give a student adequate concrete experiences from which to build mathematical insights perhaps greater use of materials such as those devised by Z. P. Dienes should be encouraged. Formalized teaching should be postponed until meaningful and varied "concrete" experiences with the basic ideas involved have been provided.

8. Group projects should be encouraged to facilitate development of the multi-perspective point of view that is considered to be essential to rational thought.

9. By releasing students from the drudgery of committing dull definitions, generalizations, and rote algorithmic procedures to memory, genuine involvement in

the building of simple mathematical systems may provide more students with an opportunity to develop positive attitudes towards mathematics and to learn to operate effectively in mathematical situations.

10. Mathematics teachers would do well to help students to dissociate ideas from original contexts and concrete origins, from their own wishes and needs, and from their own viewpoint. Ideally, a student should have an opportunity to thus arrive at abstractions, forms, and relationships that can be applied to future situations with flexibility and generality.

VII. IMPLICATIONS FOR FURTHER RESEARCH

Considering that Skemp's operations test is at least as useful a predictor of success in mathematics as any of the tests from the best available aptitudes battery, it might be fruitful to modify the test to, say, multiple choice format. This would shorten administration time and make scoring considerably less cumbersome. After validation studies using the original instrument had been carried out, the new version of the test could be used for large scale replications of the present study at various grade levels. Information gathered from such studies could be used to refine the instrument to produce an index of

mathematics potential that could be used across many grade levels. If the study were to be replicated, some scheme for inducing anxiety might be introduced (perhaps this would involve simply making the mathematics criterion a school "final") and a physical measure of anxiety (such as palmar sweating) might be administered during the testing situation to give a more accurate index of anxiety.

As for the assessment of the development of reflective intelligence, a truly longitudinal study would be desirable. After initiating the study with a grade four or five group, annual testing could be carried out with the same set of students, using Skemp's reflective intelligence tests to try to detect growth patterns. Individual as well as various sub-group growth patterns could be examined. For this purpose the modified format of the test referred to in the preceding paragraph would perhaps be preferable, to facilitate more efficient administration and greater ease of scoring.

It would be interesting and perhaps informative to carry out an experiment to assess the effects on a student's problem solving patterns that might accrue from writing Skemp's test. Does the type of reasoning that is required in writing these tests give the student insights into how to tackle problems more effectively?

Following Skemp's discussion of a method of teaching by building concepts in hierarchical order, one might attempt to analyze a school mathematics course to identify its basic concepts and operations. Then a natural hierarchy of ideas might be decided upon, and the course could be presented to students with the requisite concepts and operations being built in the natural order. The achievement of the students studying the experimental course could be compared with that of a control group studying the course in the usual sequence.

Another aspect of mathematics teaching that could be investigated within the framework of Skemp's theory would be to compare the achievement level of students subjected to a rigorous, axiomatic, definition-oriented course with that of students studying a course of comparable content but in which the ideas are built intuitively and out of "concrete" experiences. In the latter course symbolization and formalization of procedures would be deferred until the students had gained an intuitive grasp of the subject matter.

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APPENDIX

SKEMP'S TESTS

The following thirty pages constitute Skemp's SK4 and SK6 tests in the form that they were used in the present study. The format of each of the tests has been revised somewhat from that of the original, but the line drawings in each item have been faithfully copied.

The original versions of the tests had been secured from Skemp with the understanding that they were to be used only for the investigator's personal research. If the reader should wish to make use of these tests or the originals, he should seek permission from Dr. R. R. Skemp, Lecturer in Psychology, Manchester University, England.

Skemp's doctoral dissertation is available on microfilm in the library of the University of Alberta.

In the present study, the SK4 test was administered during one class period and the SK6 test during another. SK4, Parts I and II, were distributed to the students in a single booklet. The instructions on the cover sheet were read and the students were given eight minutes to complete Part I. Most students were able to finish within the eight minute time limit, and they were asked to wait for instructions to begin Part II. After time had been called for Part I, the instructions for Part II were read aloud and the students were given twenty minutes to complete the test.

When the SK6 test was administered, the Practice Sheet, the Demonstration Sheet, and Part I were distributed, and the practice questions were worked through with the class. Then the Demonstration Sheet was explained, and the students were given eight minutes to complete Part I. After the students' responses had been collected, the Answer Sheet for SK6, Part I was distributed along with SK6, Part II. The answers to Part I were discussed and student inquiries were answered to ensure that the basic operations would be understood. The procedure for SK6, Part II was explained and the students were given twenty minutes to complete the test.

NAME .... SCHOOL ....
Last First Middle

AGE .... Years GRADE .... BOY GIRL (Circle One) DATE .... Day Month Year

SK4: PART I

INSTRUCTIONS

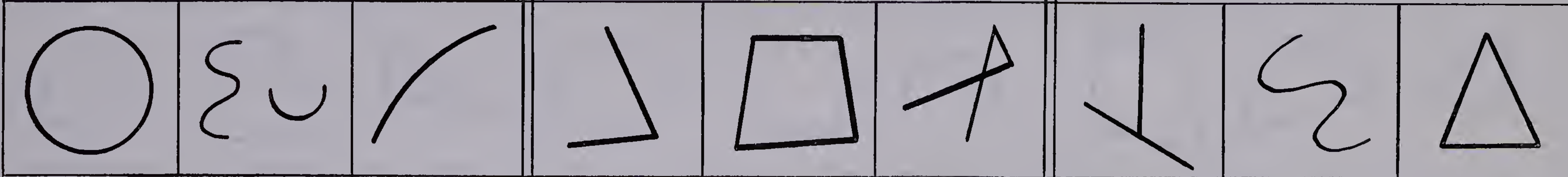
There are 15 rows of figures in PART I of this test. In each row of figures, the first three figures (marked "EXAMPLES") all have some property in common. In the second group of three (marked "NOT EXAMPLES") in the row, none of the figures has this property. Your problem is to decide whether or not each of the figures under the question "ARE THESE EXAMPLES?" has the property. If it has the property (i.e., if it is an example), circle YES under that figure. If it does not have the property, circle NO.

Look at the first row of figures at the top of the next page. Each of the figures under "EXAMPLES" is made of curved lines only. None of the figures under "NOT EXAMPLES" has this property. Look under the question "ARE THESE EXAMPLES?" Figure 1 is not made of curved lines, so you should circle NO for figure 1. Since figure 2 is curved, you should circle YES for figure 2. Figure 3 is not made of curved lines so you should circle NO for figure 3.

The second row of figures deals with another property. After deciding what the property is by looking at the figures under "EXAMPLES" and under "NOT EXAMPLES," answer the question "ARE THESE EXAMPLES?" by circling YES or NO for figure 4; then for figure 5; and then for figure 6. Then go on to the rest of the rows of figures in PART I of the test.

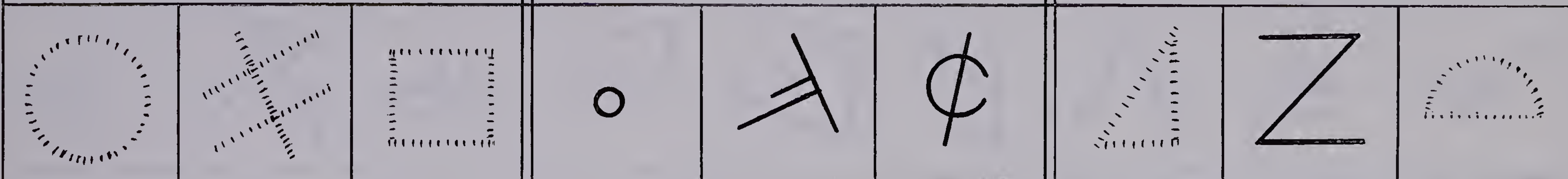
IMPORTANT: One, two, or all three of the figures under "ARE THESE EXAMPLES?" may have the property (i.e., may be examples).

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



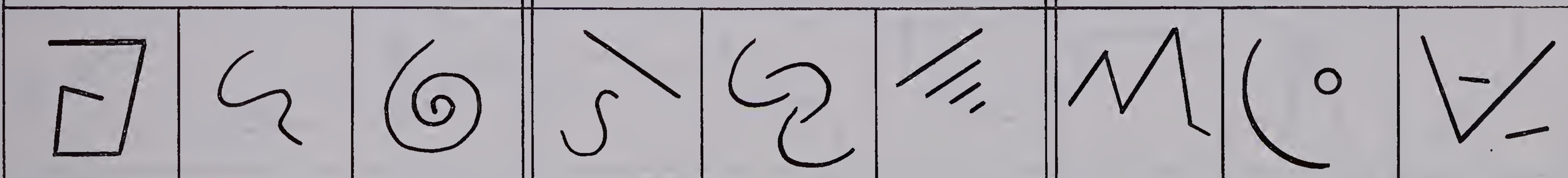
1. YES NO
2. YES NO
3. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



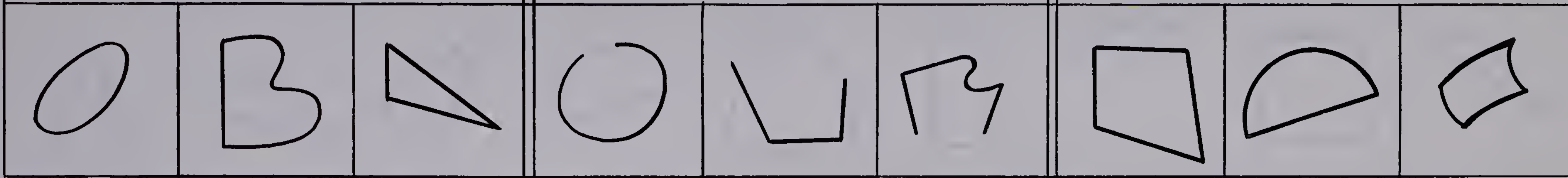
4. YES NO
5. YES NO
6. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



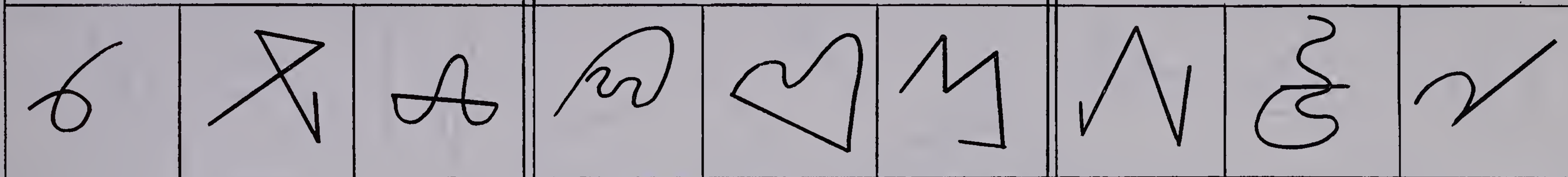
7. YES NO
8. YES NO
9. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



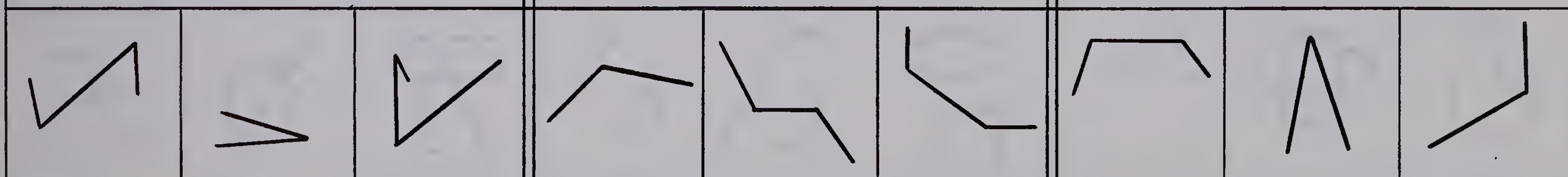
10. YES NO	11. YES NO	12. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



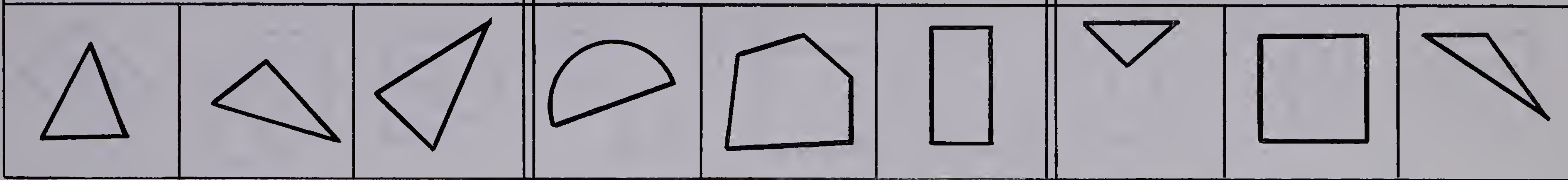
13. YES NO	14. YES NO	15. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



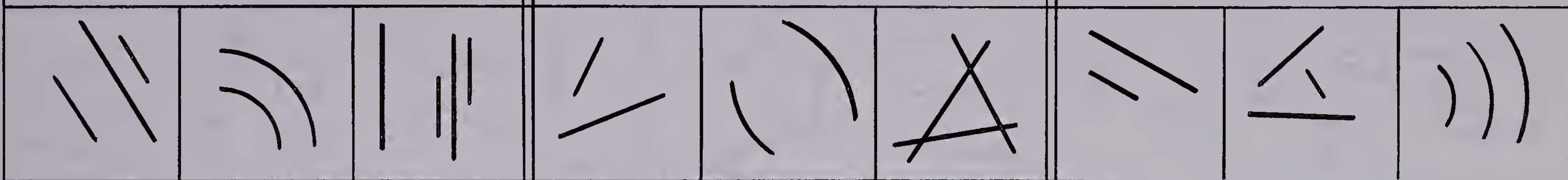
16. YES NO	17. YES NO	18. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



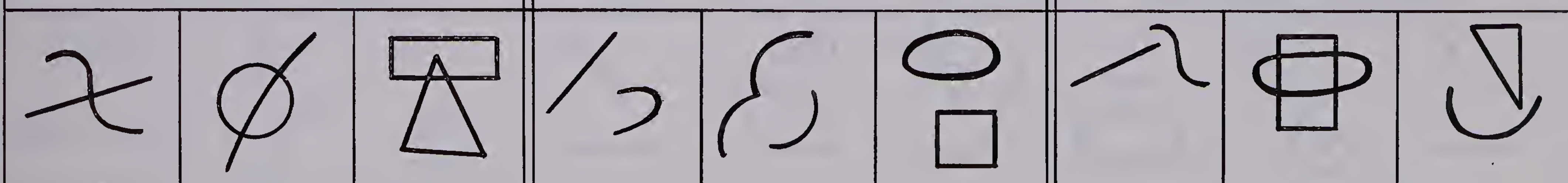
19. YES NO	20. YES NO	21. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



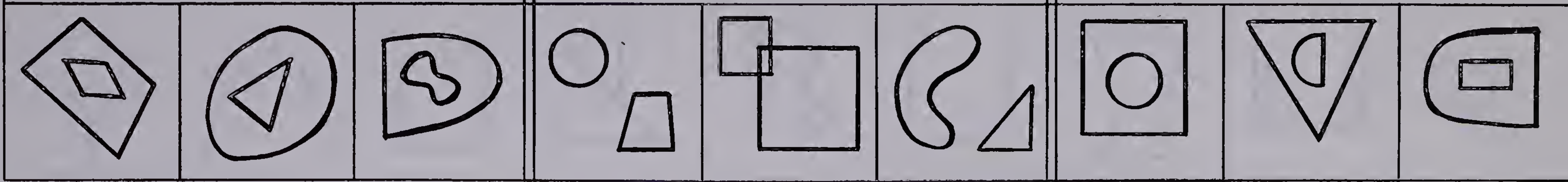
22. YES NO	23. YES NO	24. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



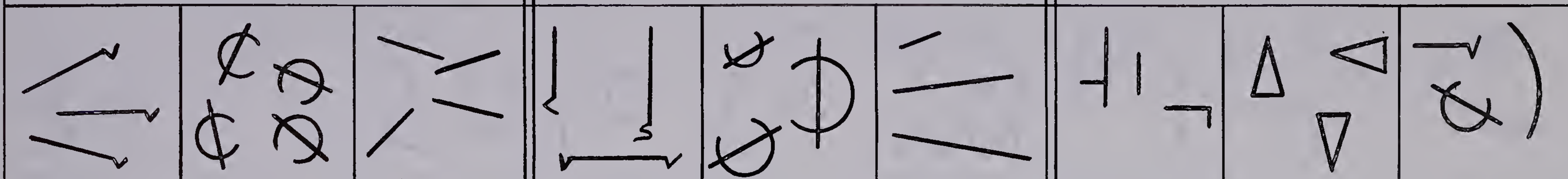
25. YES NO	26. YES NO	27. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



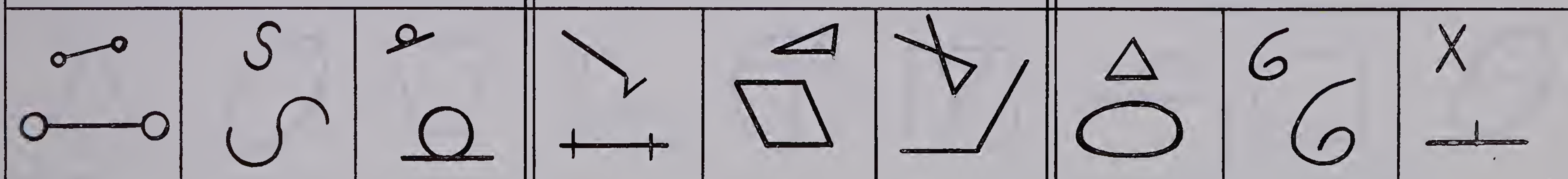
28. YES NO
29. YES NO
30. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?

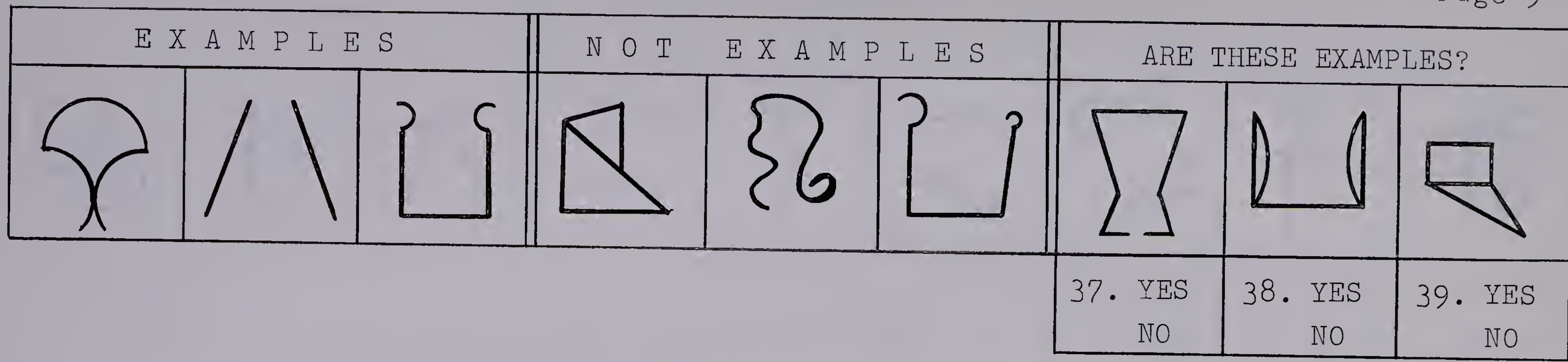


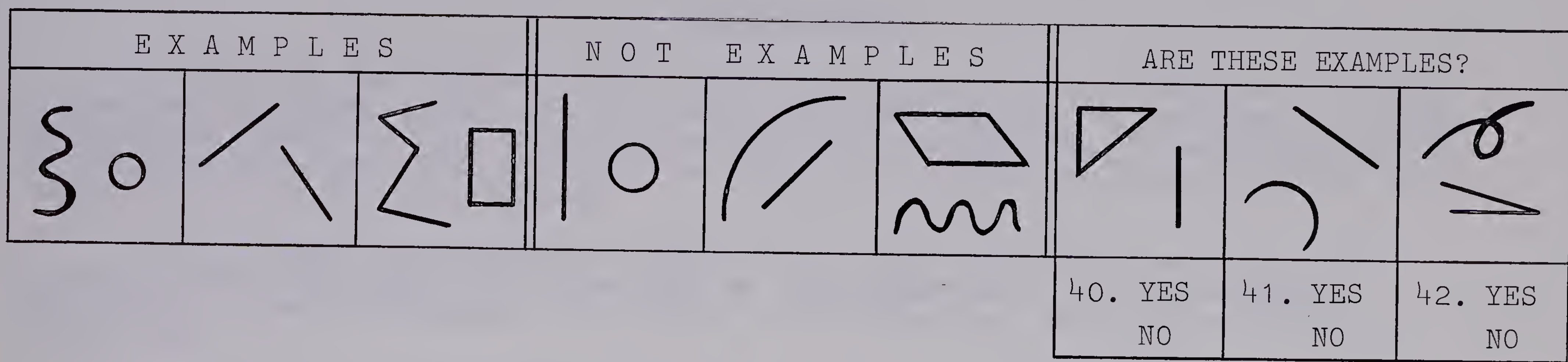
31. YES NO
32. YES NO
33. YES NO

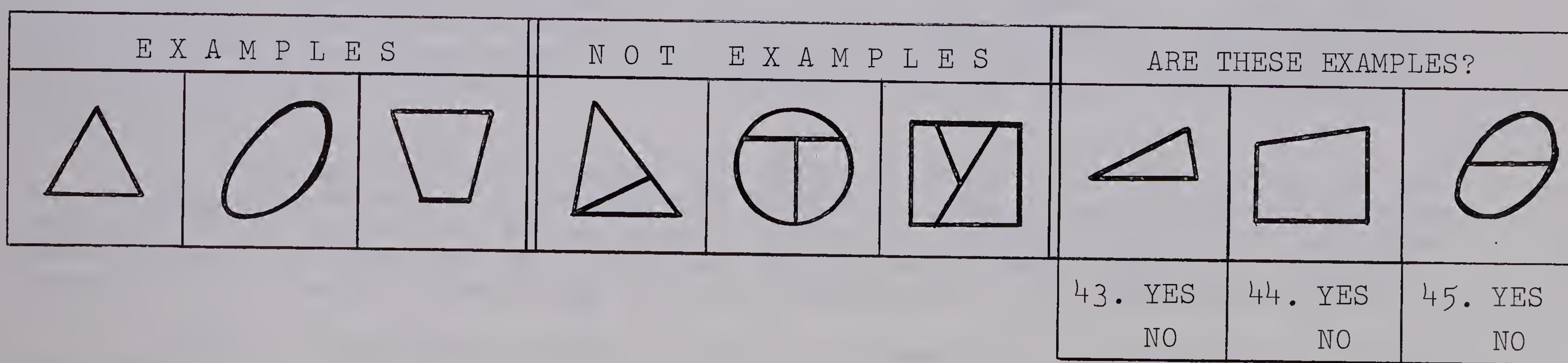
EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



34. YES NO
35. YES NO
36. YES NO


E X A M P L E S
N O T E X A M P L E S
ARE THESE EXAMPLES?
37. YES
NO
38. YES
NO
39. YES
NO

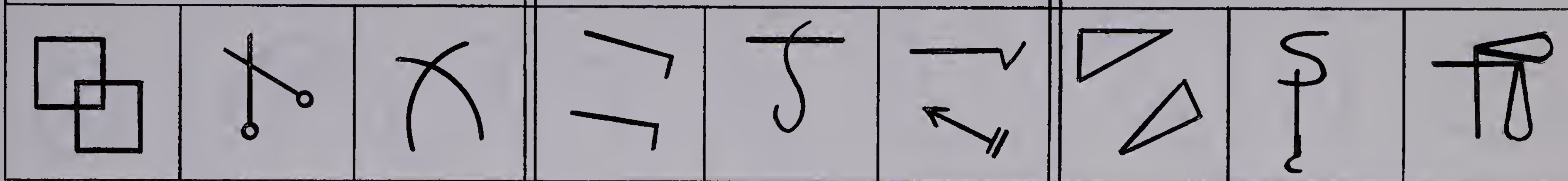

E X A M P L E S
N O T E X A M P L E S
ARE THESE EXAMPLES?
40. YES
NO
41. YES
NO
42. YES
NO


E X A M P L E S
N O T E X A M P L E S
ARE THESE EXAMPLES?
43. YES
NO
44. YES
NO
45. YES
NO

SK4: PART II

PRACTICE PROBLEM

E X A M P L E S	N O T E X A M P L E S	ARE THESE EXAMPLES?



a. YES NO	b. YES NO	c. YES NO

INSTRUCTIONS

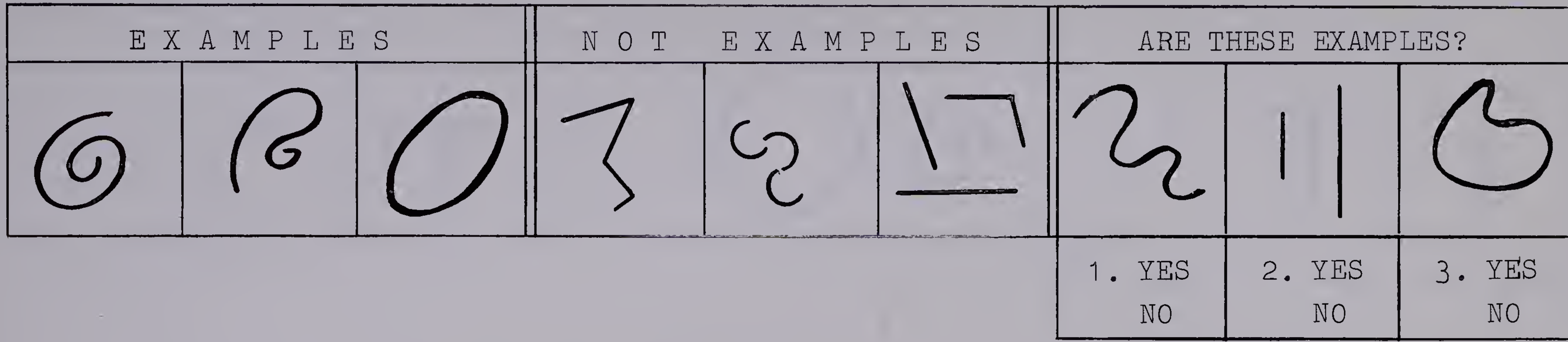
In each of the problems in PART II, the figures marked "EXAMPLES" have two properties in common. The figures marked "NOT EXAMPLES" do not have both of these properties. They may have only one, or neither. To help you, in each problem (row of figures) the first "NOT EXAMPLE" has one of the properties, the second has the other, and the third has neither.

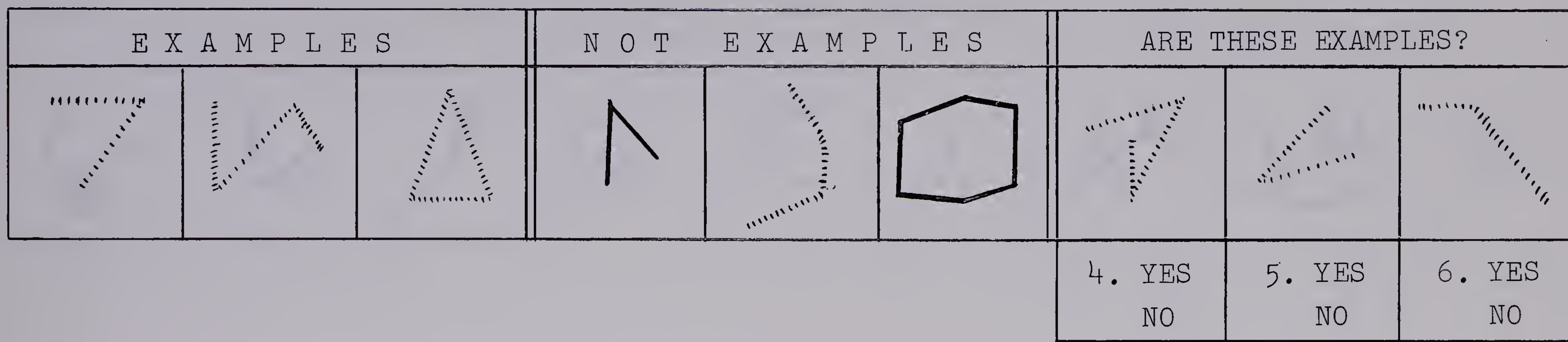
Your problem is to decide whether or not each of the figures under the question "ARE THESE EXAMPLES?" has both of the properties. If it has both properties, circle YES under that figure. If it does not have both properties, circle NO.

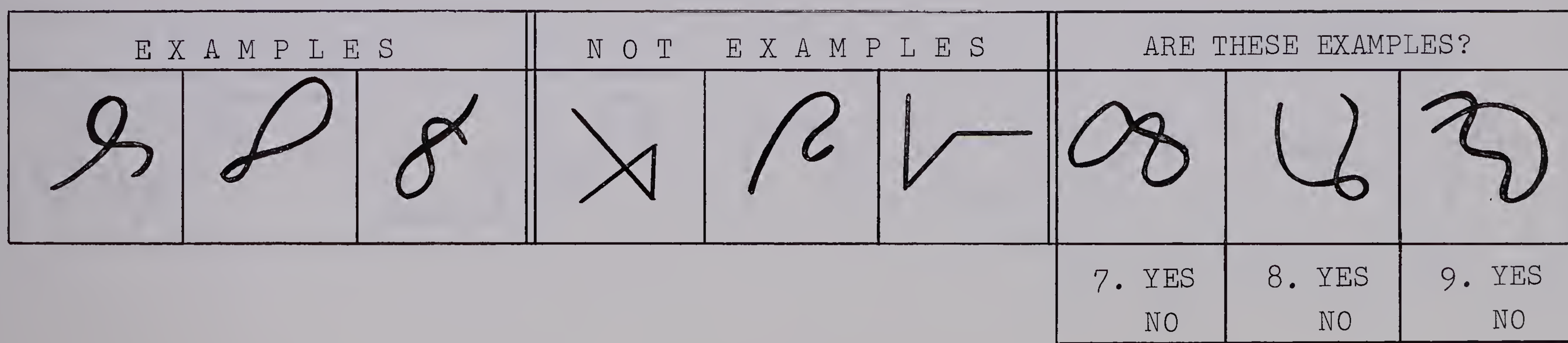
Look at the PRACTICE PROBLEM above. Each of the "EXAMPLES" is made up of two identical parts which intersect. In the first "NOT EXAMPLE," the parts are identical but do not intersect. In the second "NOT EXAMPLE," they intersect but are not identical. In the third "NOT EXAMPLE," they are neither identical nor intersecting.

For each of the figures under "ARE THESE EXAMPLES?", decide whether or not it has both properties. In the PRACTICE PROBLEM, figure a. has two identical parts but they do not intersect, so you should circle NO. Figure b. does have two intersecting parts but they are not identical, so circle NO under figure b. Figure c. does have two identical parts and they do intersect so circle YES under figure c.

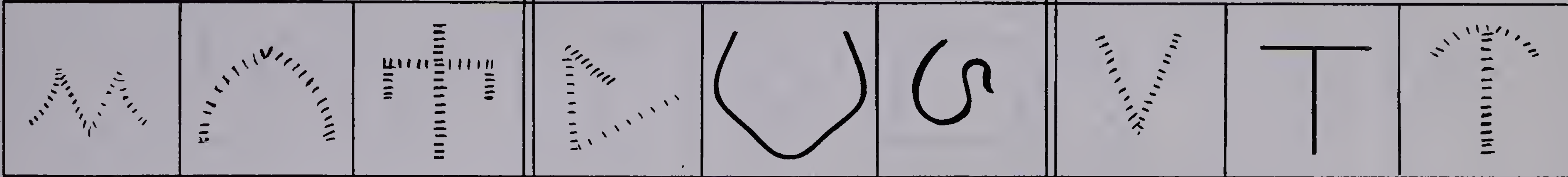
All of the problems in PART II are of the same kind as the PRACTICE PROBLEM, BUT one, two, or all three of the figures under "ARE THESE EXAMPLES?" may be examples.


E X A M P L E S
N O T   E X A M P L E S
ARE THESE EXAMPLES?
1. YES
NO
2. YES
NO
3. YES
NO


E X A M P L E S
N O T   E X A M P L E S
ARE THESE EXAMPLES?
4. YES
NO
5. YES
NO
6. YES
NO

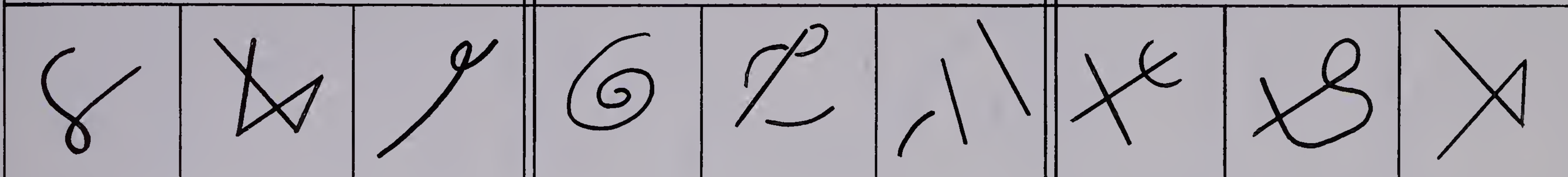

E X A M P L E S
N O T   E X A M P L E S
ARE THESE EXAMPLES?
7. YES
NO
8. YES
NO
9. YES
NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



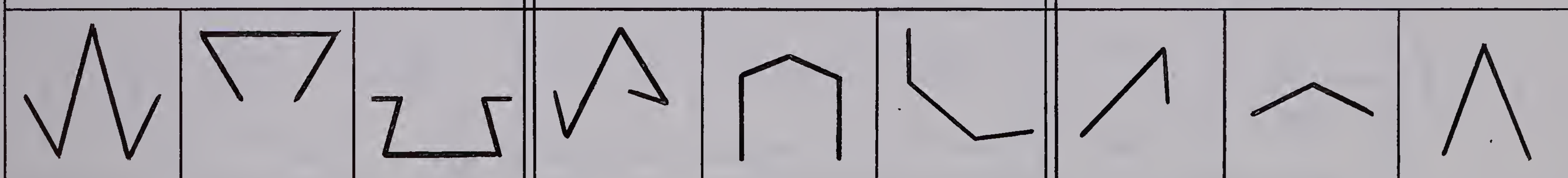
10. YES NO	11. YES NO	12. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



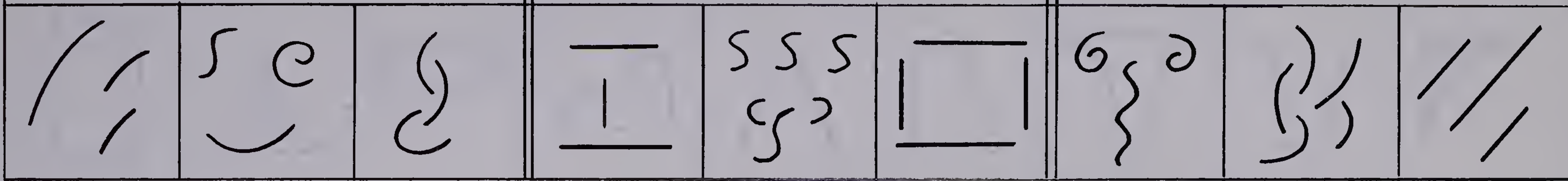
13. YES NO	14. YES NO	15. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



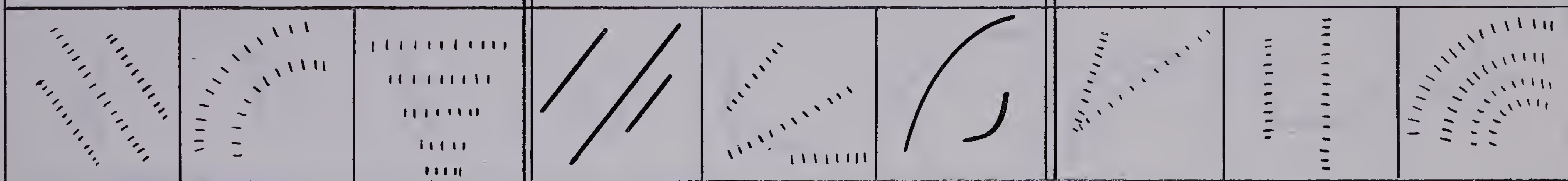
16. YES NO	17. YES NO	18. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



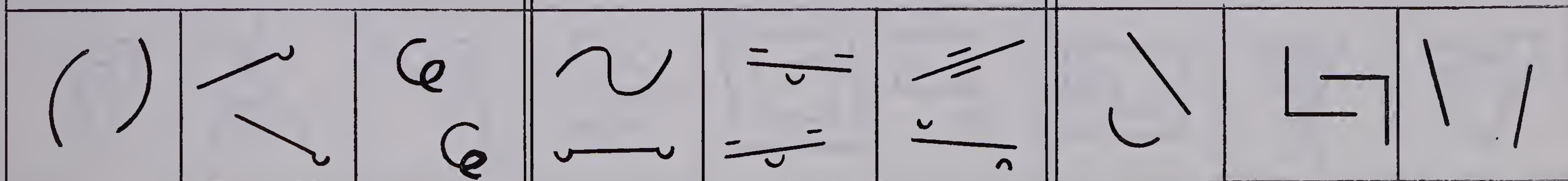
19. YES NO	20. YES NO	21. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



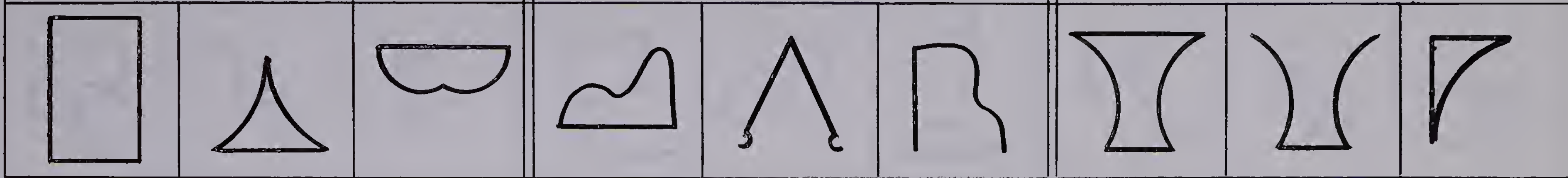
22. YES NO	23. YES NO	24. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



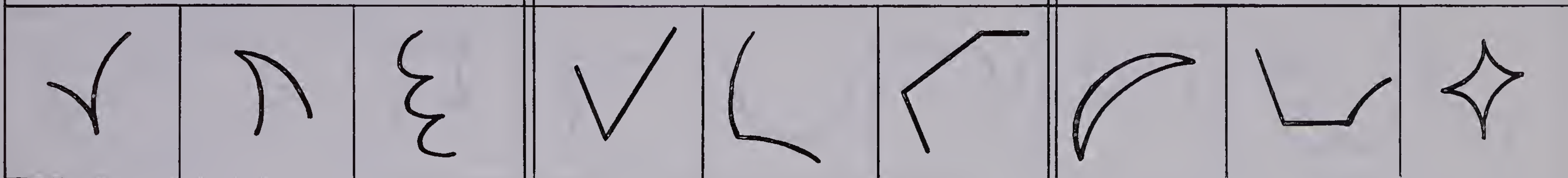
25. YES NO	26. YES NO	27. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



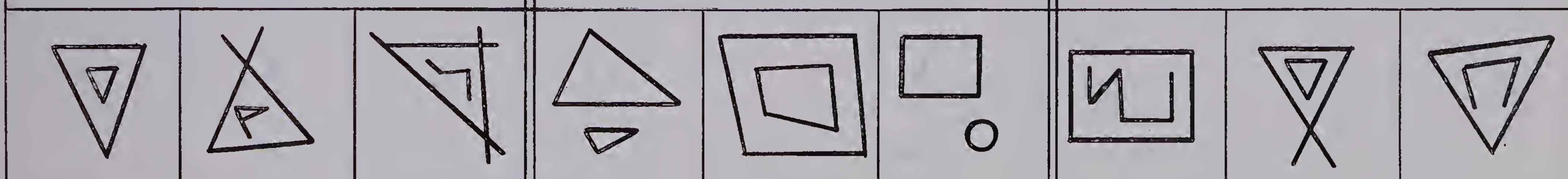
28. YES NO	29. YES NO	30. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



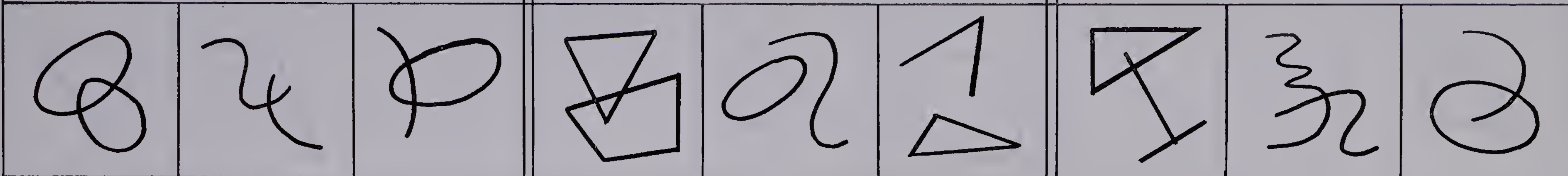
31. YES NO	32. YES NO	33. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



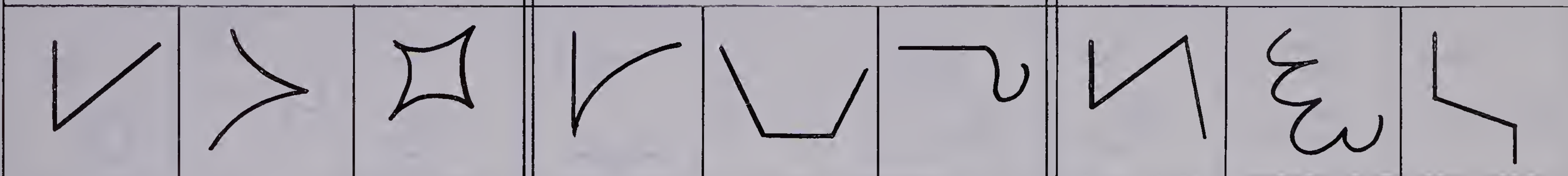
34. YES NO	35. YES NO	36. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



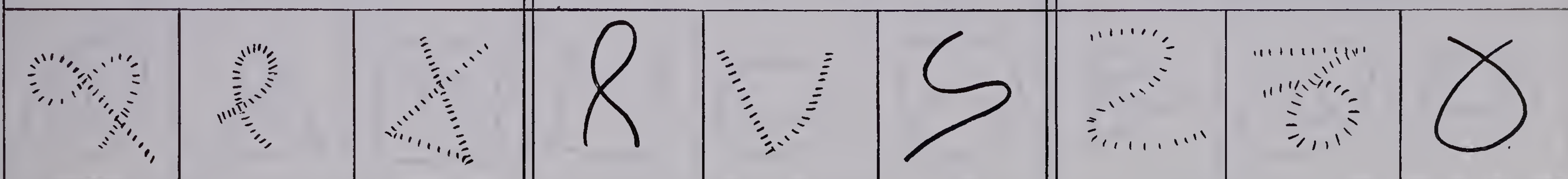
37. YES NO
38. YES NO
39. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



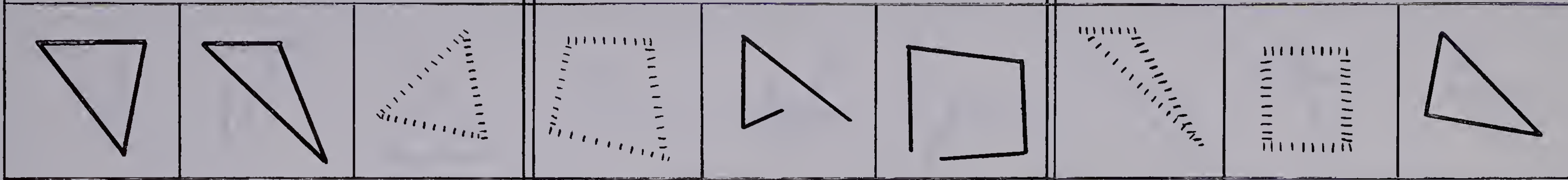
40. YES NO
41. YES NO
42. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



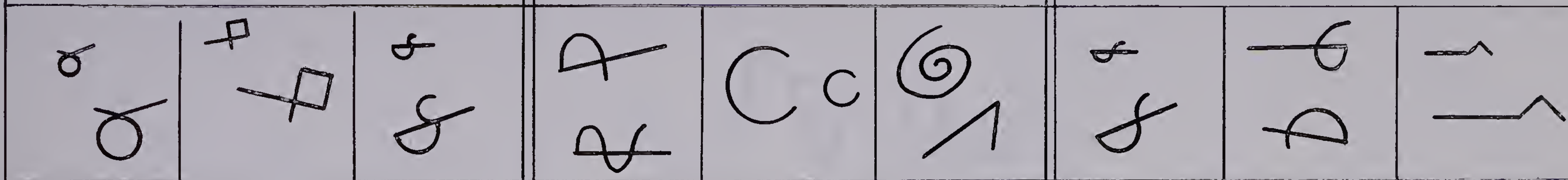
43. YES NO
44. YES NO
45. YES NO

E X A M P L E S	N O T E X A M P L E S	ARE THESE EXAMPLES?



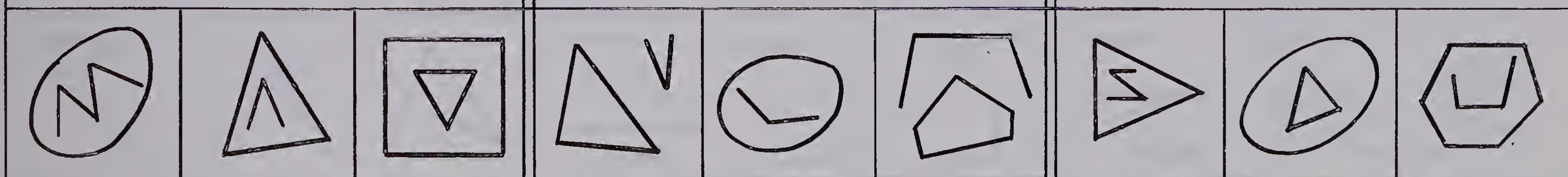
46. YES NO	47. YES NO	48. YES NO

E X A M P L E S	N O T E X A M P L E S	ARE THESE EXAMPLES?

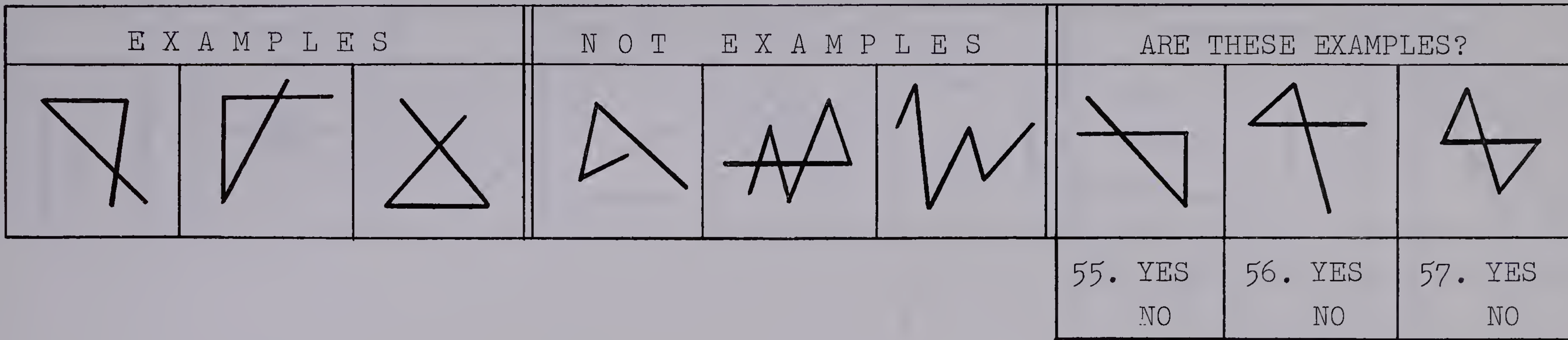


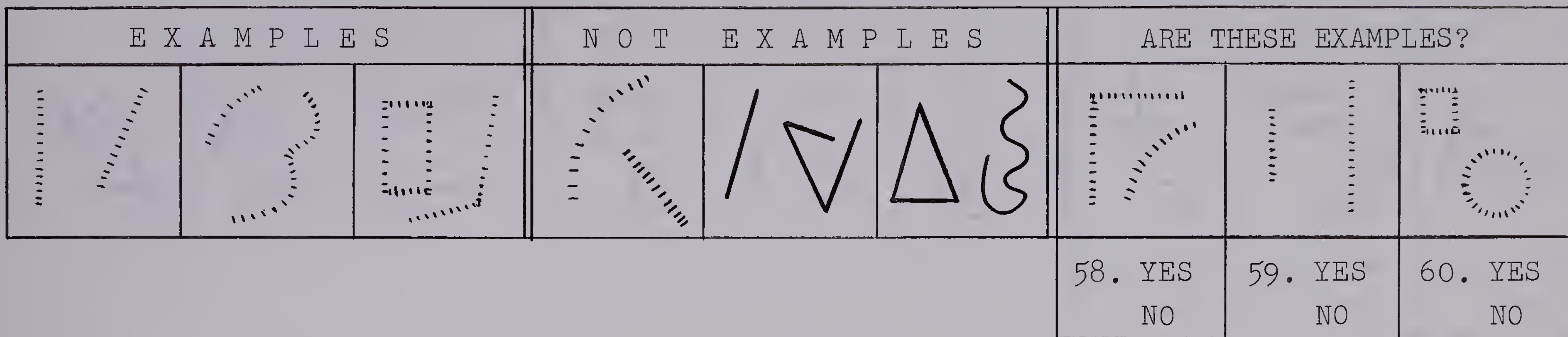
49. YES NO	50. YES NO	51. YES NO

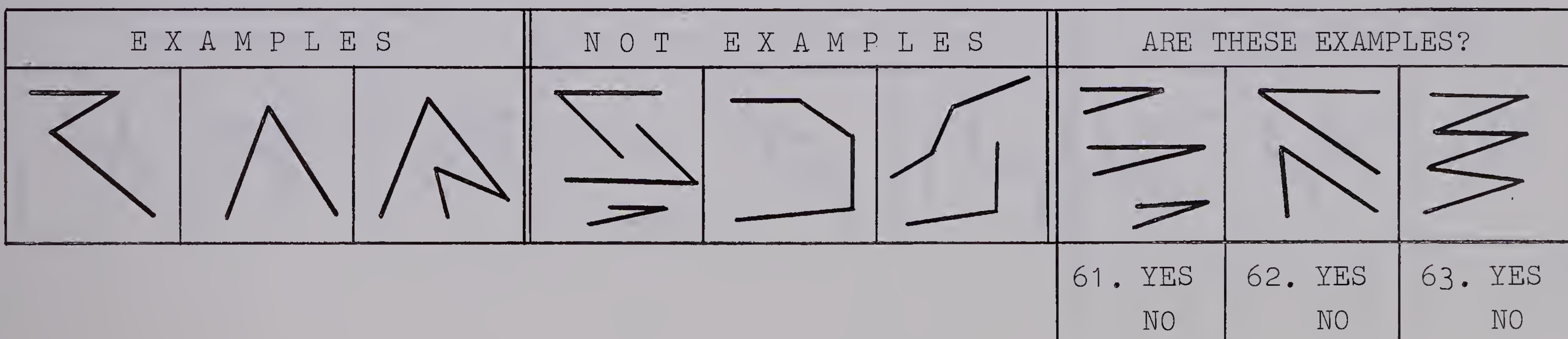
E X A M P L E S	N O T E X A M P L E S	ARE THESE EXAMPLES?

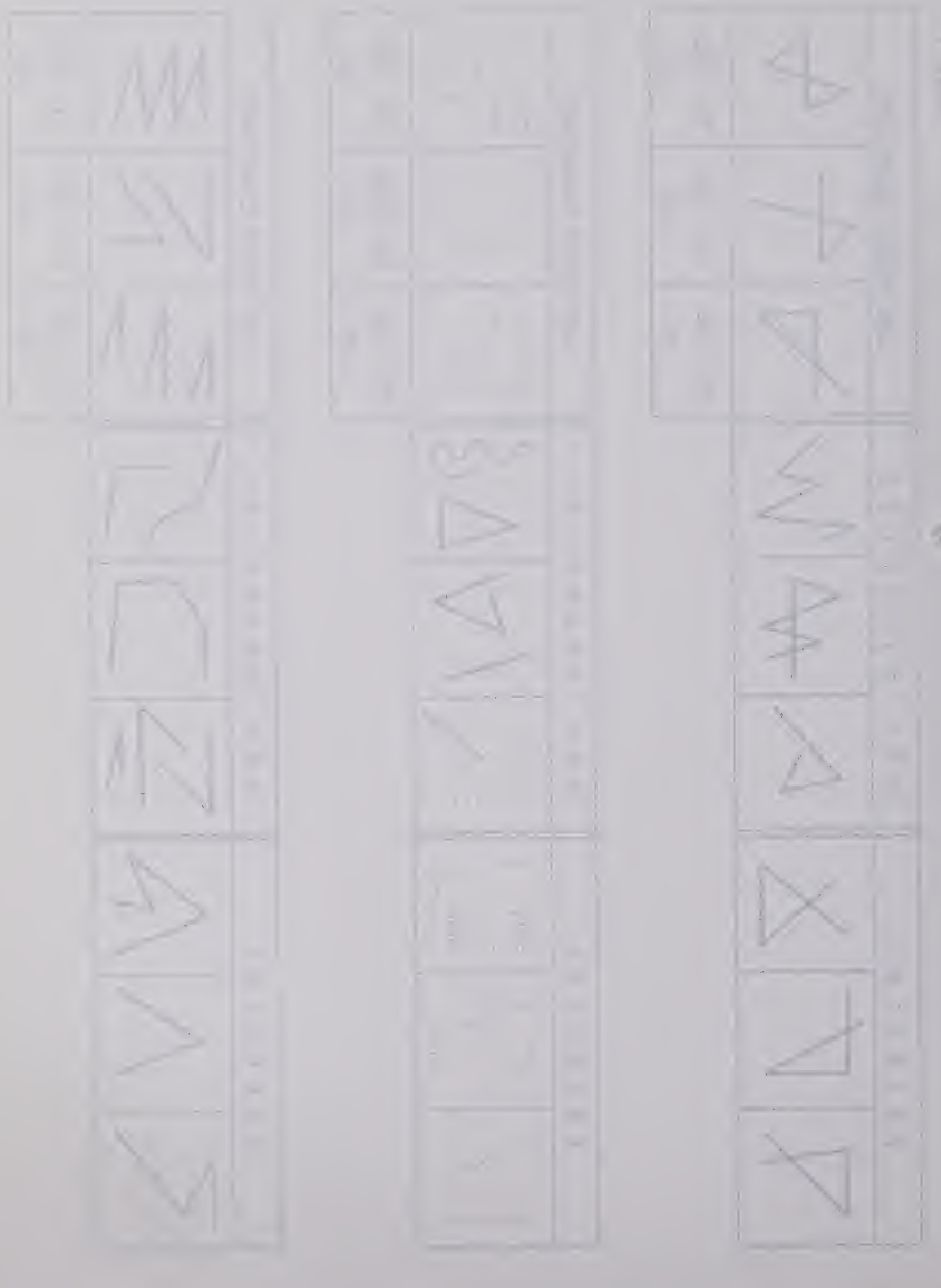


52. YES NO	53. YES NO	54. YES NO

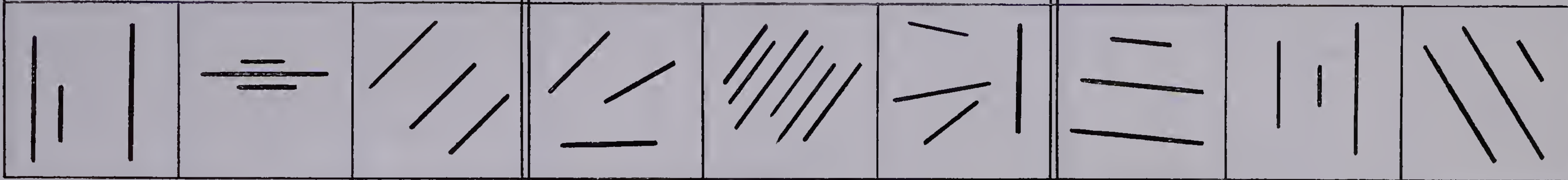

EXAMPLES
NOT EXAMPLES
ARE THESE EXAMPLES?
55. YES NO
56. YES NO
57. YES NO


EXAMPLES
NOT EXAMPLES
ARE THESE EXAMPLES?
58. YES NO
59. YES NO
60. YES NO


EXAMPLES
NOT EXAMPLES
ARE THESE EXAMPLES?
61. YES NO
62. YES NO
63. YES NO

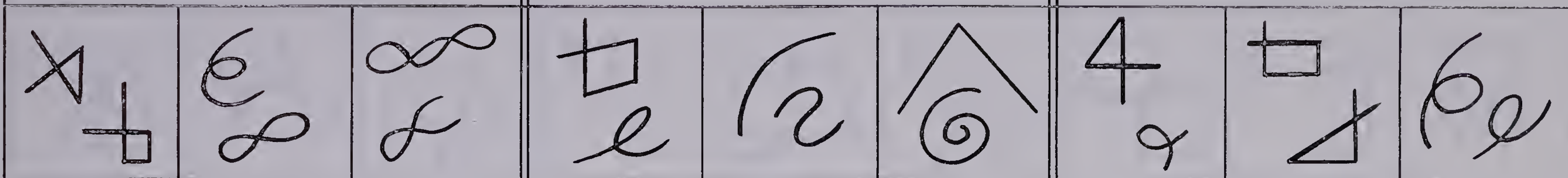


EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



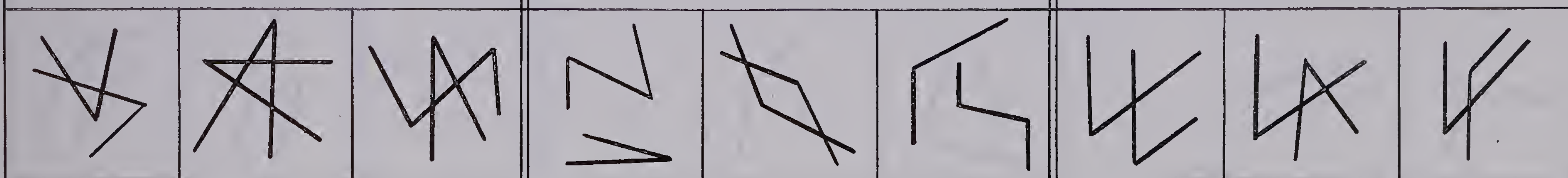
64. YES NO	65. YES NO	66. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?

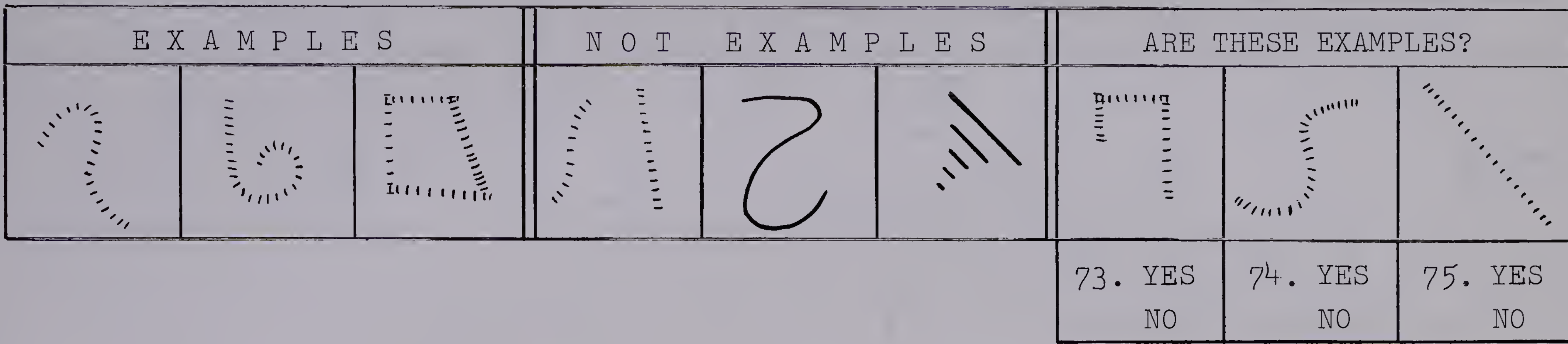


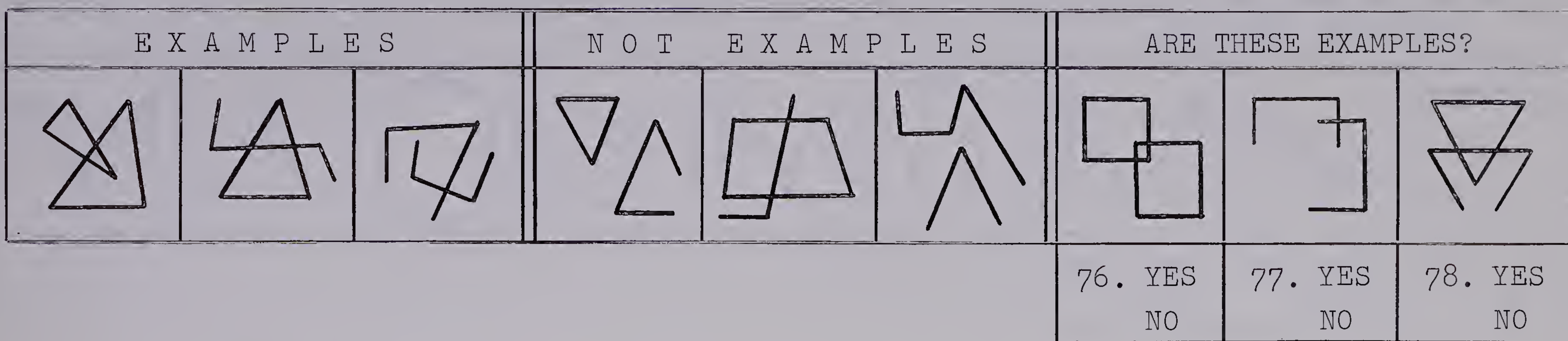
67. YES NO	68. YES NO	69. YES NO

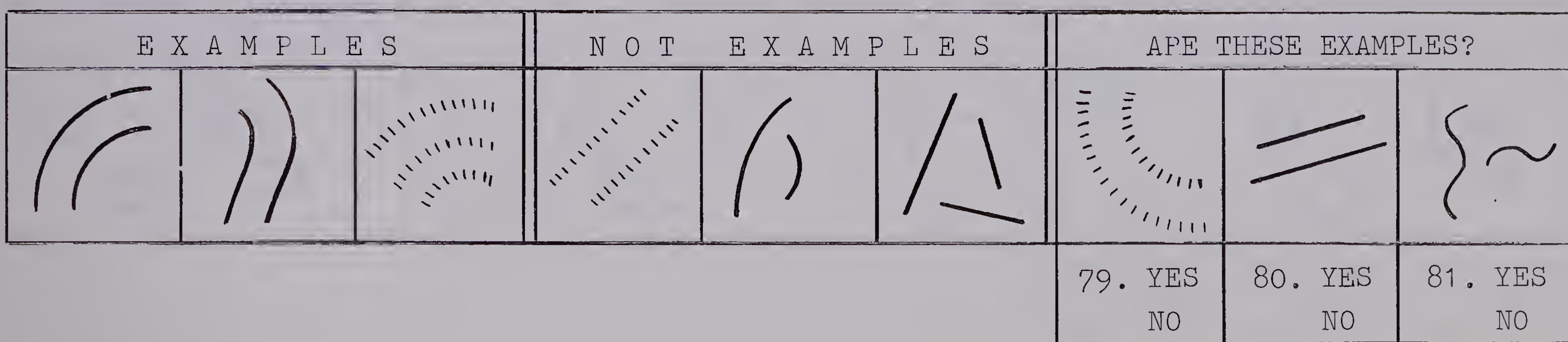
EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?

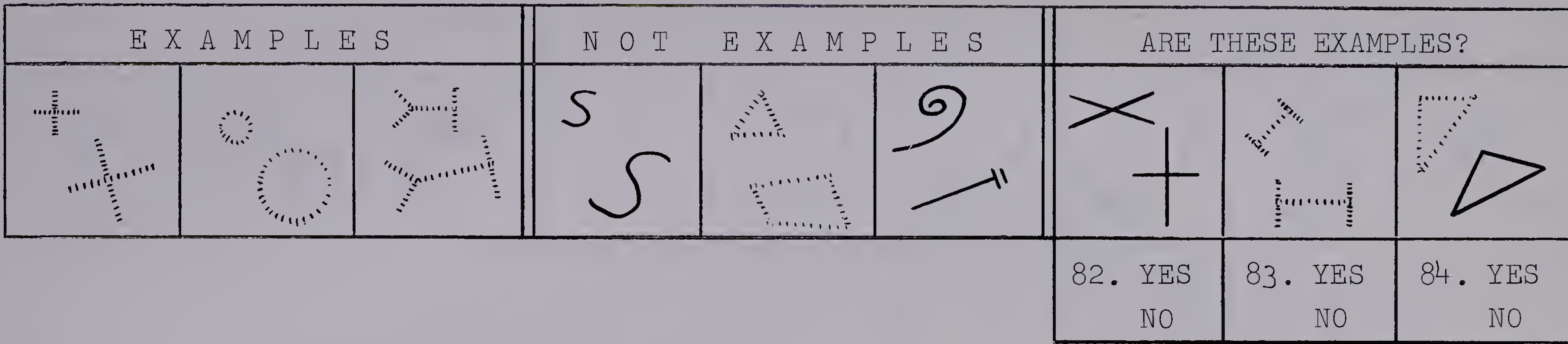


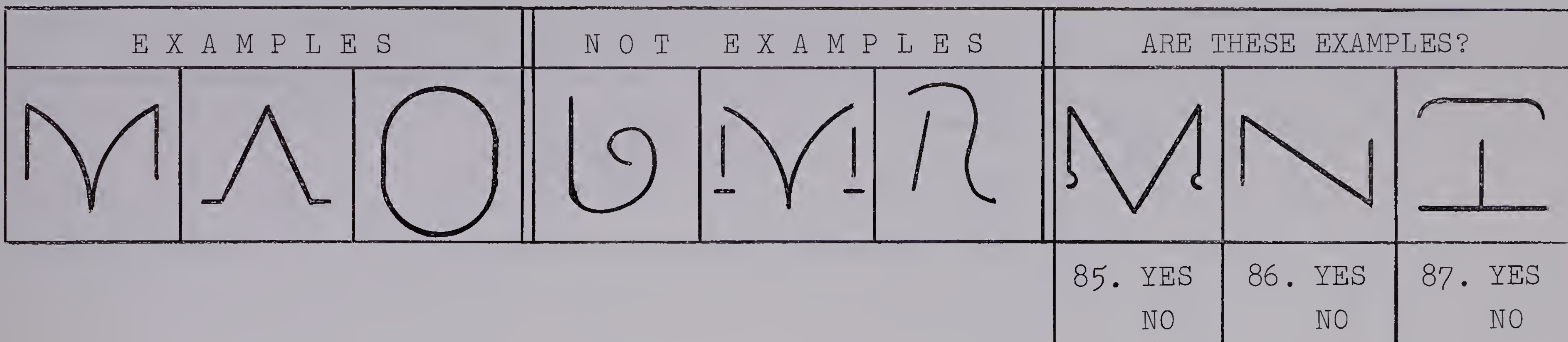
70. YES NO	71. YES NO	72. YES NO

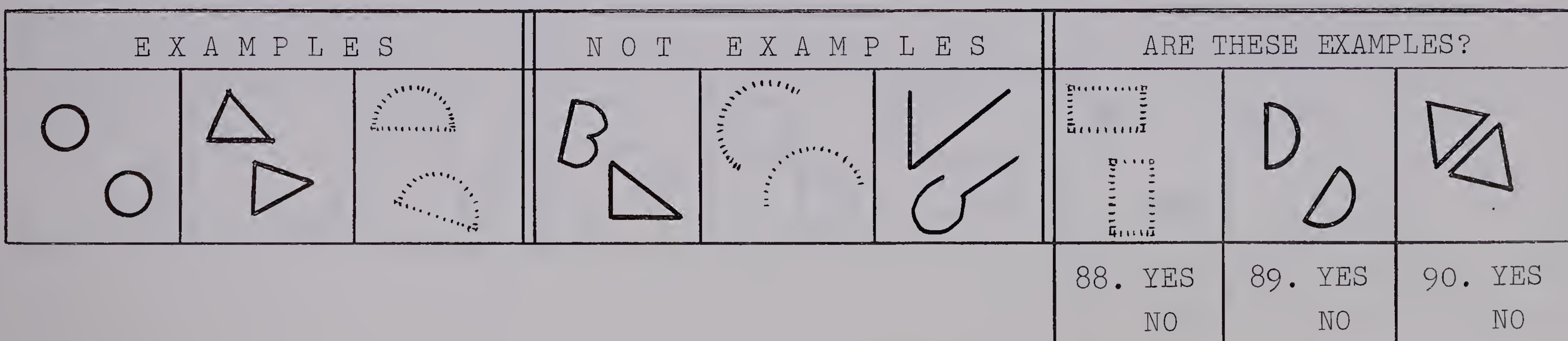

EXAMPLES
NOT EXAMPLES
ARE THESE EXAMPLES?
73. YES NO
74. YES NO
75. YES NO


EXAMPLES
NOT EXAMPLES
ARE THESE EXAMPLES?
76. YES NO
77. YES NO
78. YES NO

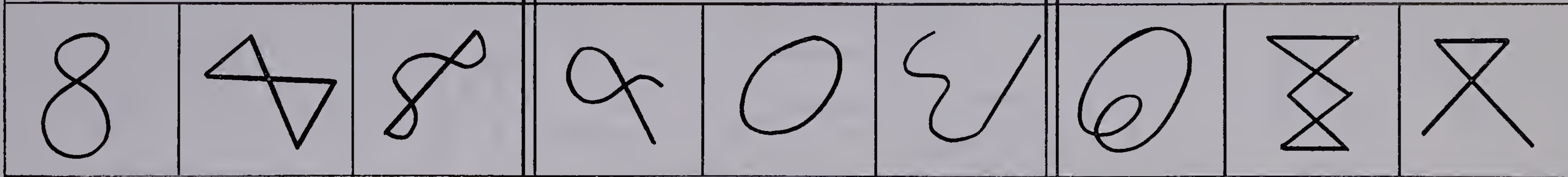

EXAMPLES
NOT EXAMPLES
APE THESE EXAMPLES?
79. YES NO
80. YES NO
81. YES NO


EXAMPLES
NOT EXAMPLES
ARE THESE EXAMPLES?
82. YES
NO
83. YES
NO
84. YES
NO


EXAMPLES
NOT EXAMPLES
ARE THESE EXAMPLES?
85. YES
NO
86. YES
NO
87. YES
NO

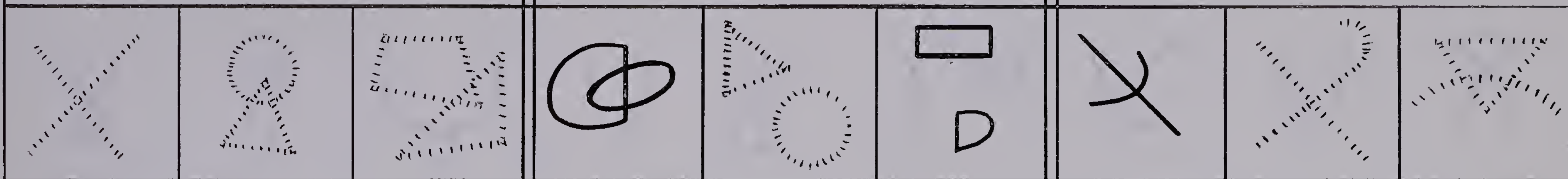

EXAMPLES
NOT EXAMPLES
ARE THESE EXAMPLES?
88. YES
NO
89. YES
NO
90. YES
NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



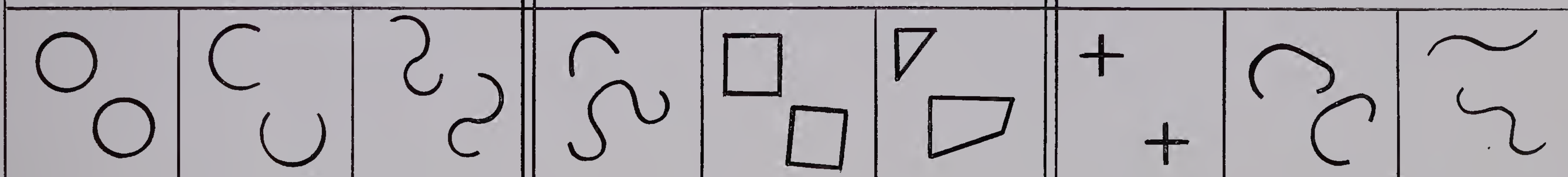
91. YES NO	92. YES NO	93. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



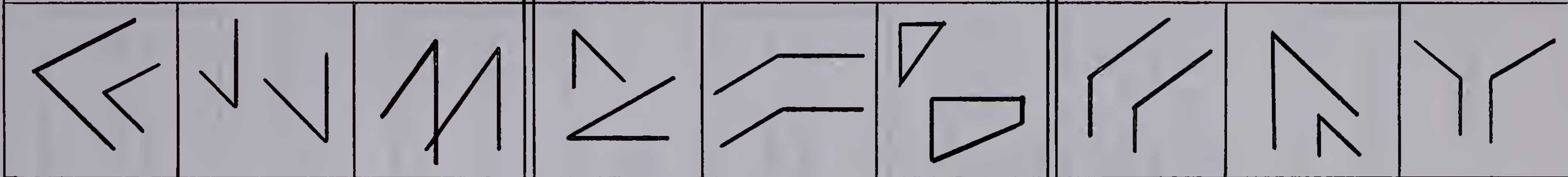
94. YES NO	95. YES NO	96. YES NO

EXAMPLES	NOT EXAMPLES	ARE THESE EXAMPLES?



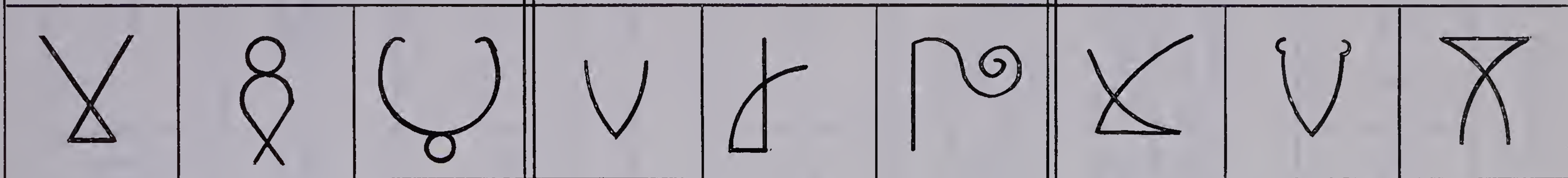
97. YES NO	98. YES NO	99. YES NO

E X A M P L E S	N O T E X A M P L E S	ARE THESE EXAMPLES?



100. YES NO	101. YES NO	102. YES NO

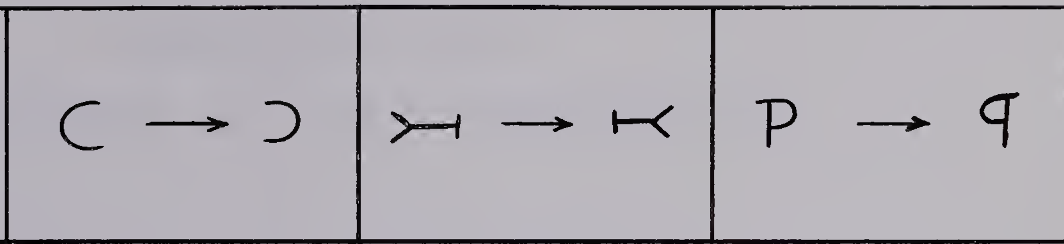
E X A M P L E S	N O T E X A M P L E S	ARE THESE EXAMPLES?



103. YES NO	104. YES NO	105. YES NO

SK6: PRACTICE SHEET

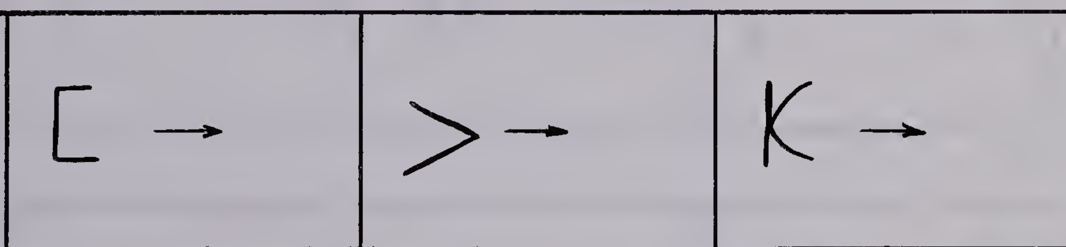
Operation 1



In the above figures, the one on the left of each pair has been changed to the one on the right by means of the same simple operation. In other words, the above figures give three examples of a particular operation. You have to find out what the operation is, and then do the same operation to some other figures.

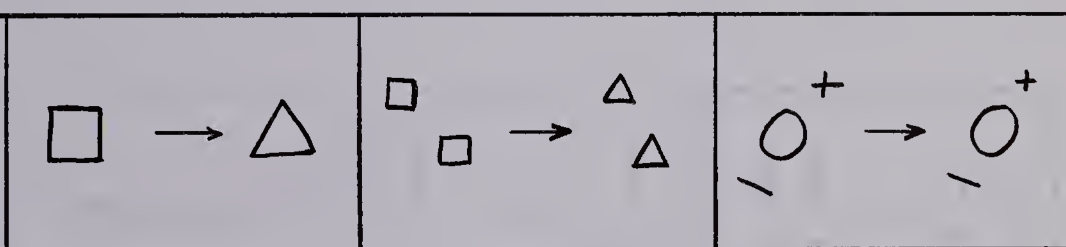
What is the operation? It is reversing from left to right. Do this on each of the figures below, and fill in the answers in the blank spaces. Check with the answers on the blackboard to make sure that you have understood.

Do Operation 1 on these.



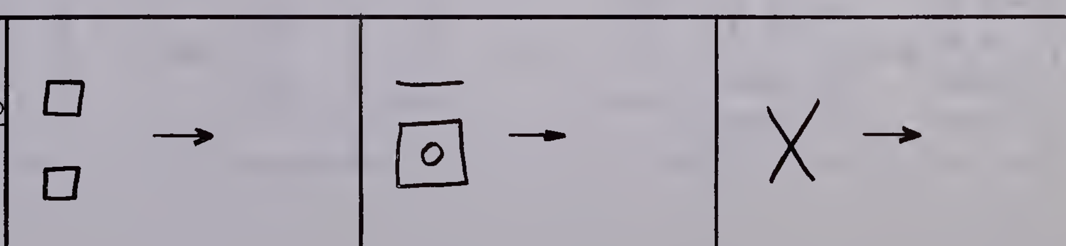
Here is a different operation:

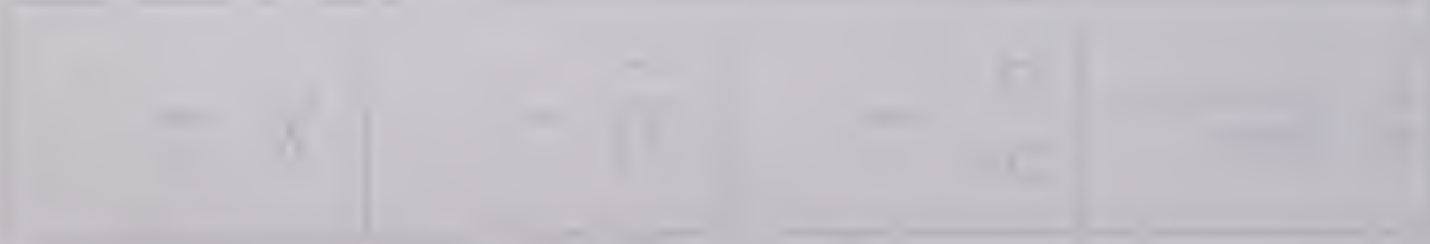
Operation 2



When you have found out what it is, do it on the figures below. Check with the answers on the board.

Do Operation 2 on these.



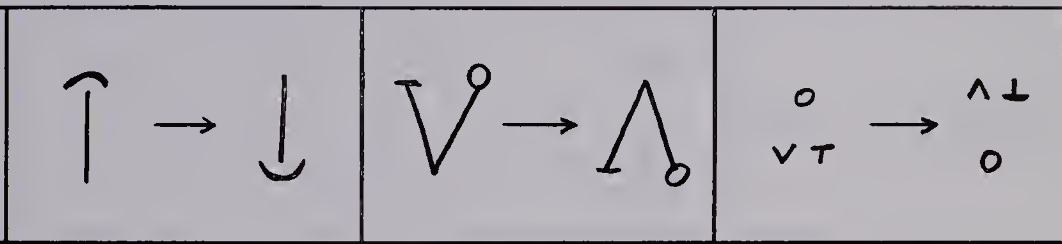


SK6: DEMONSTRATION SHEET

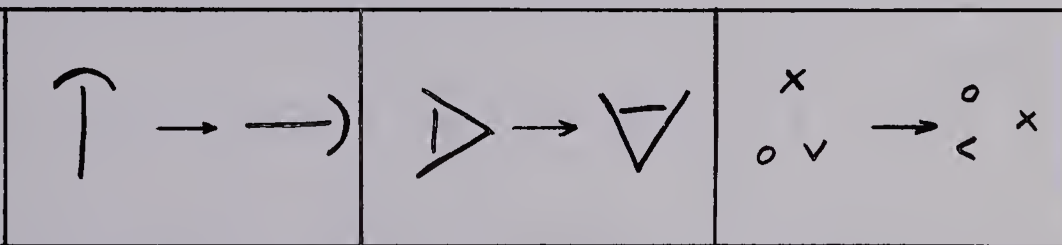
OPERATIONS A TO E

(OPERATIONS F TO J ARE ON THE NEXT PAGE)

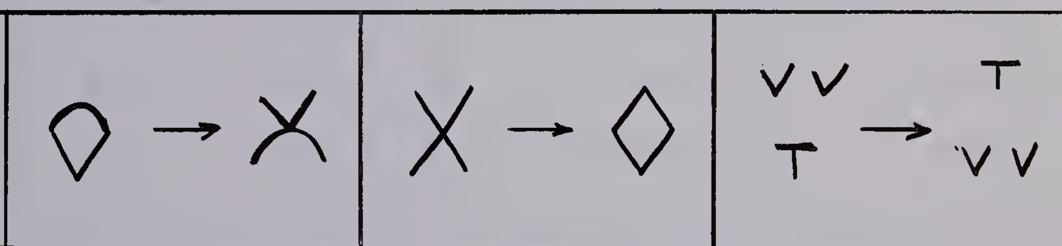
Operation A



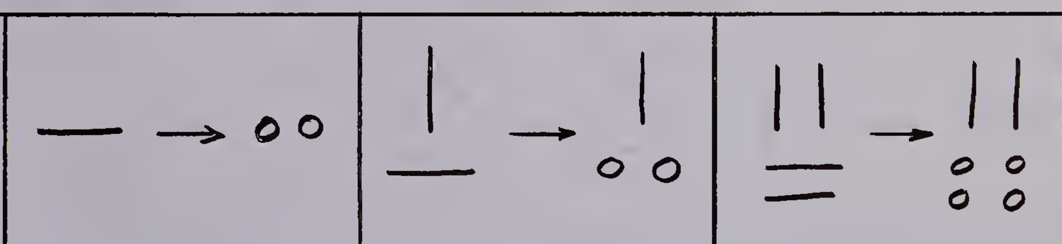
Operation B



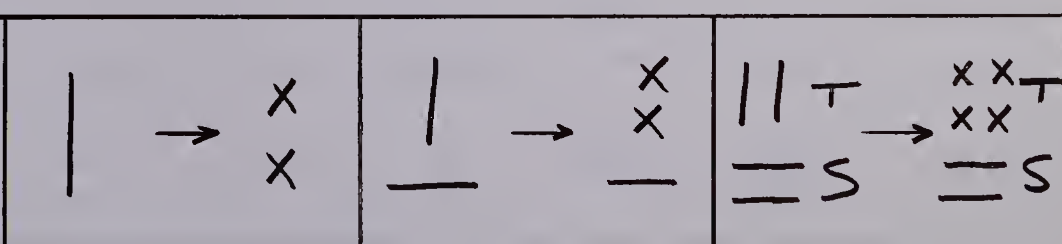
Operation C



Operation D

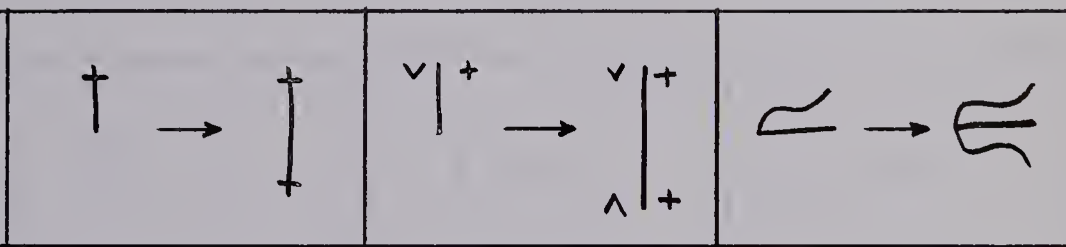
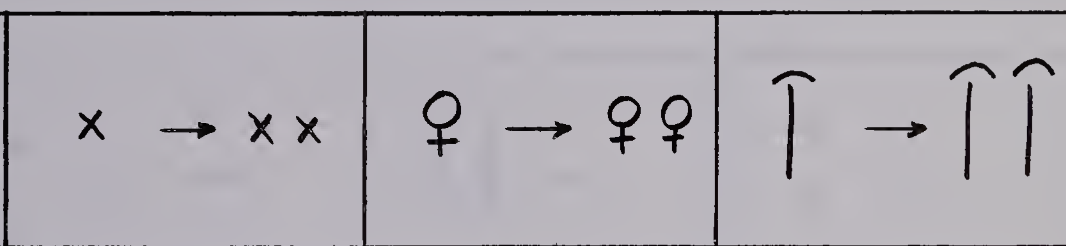
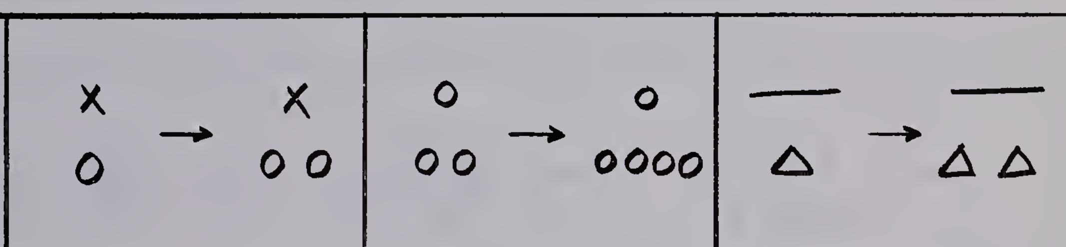
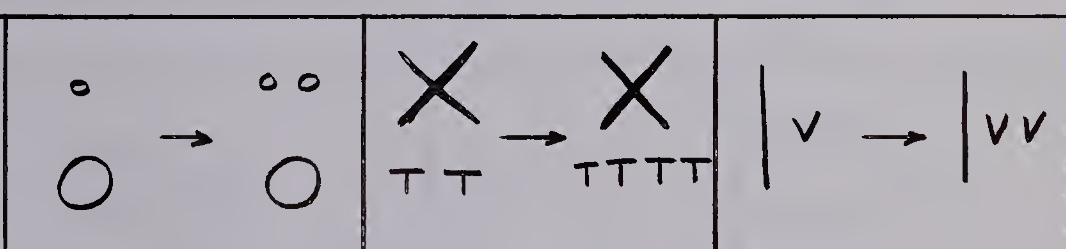
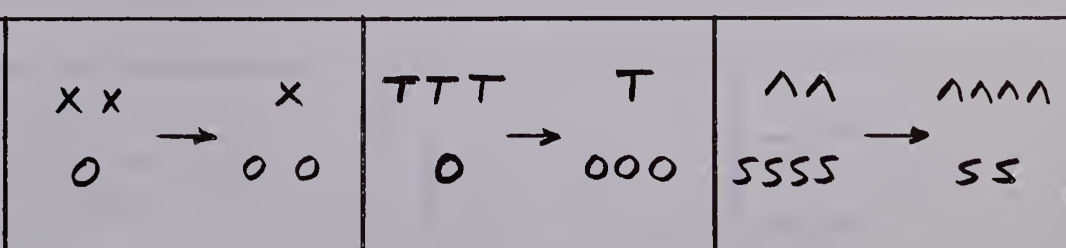


Operation E



SK6: DEMONSTRATION SHEET

OPERATIONS F TO J

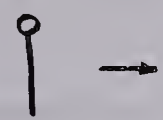
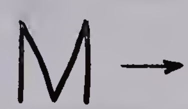
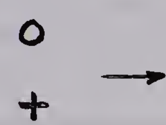
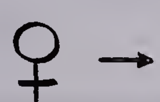
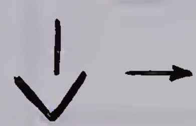
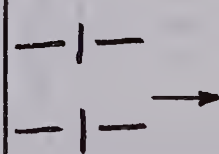
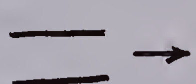
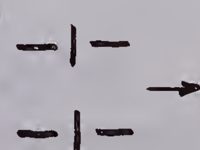
Operation F	
Operation G	
Operation H	
Operation I	
Operation J	

NAME .... SCHOOL ....
Last First Middle

AGE .... GRADE .... BOY GIRL DATE ....
Years (Circle One) Day Month Yr.

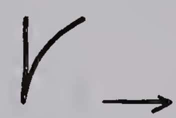
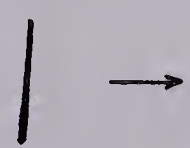
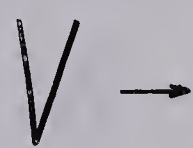
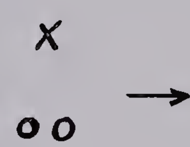
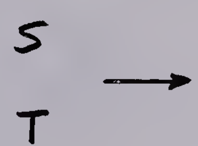
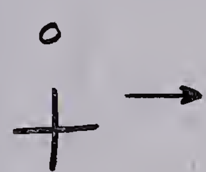
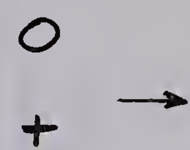
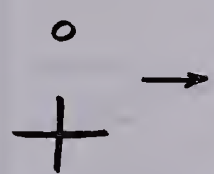
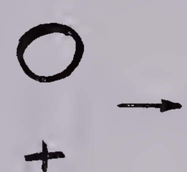
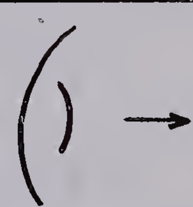
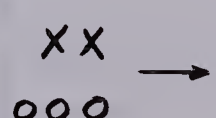
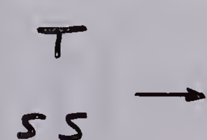
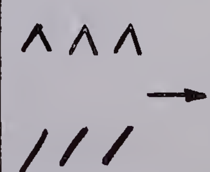
SK6: PART I

Find out the operations from the DEMONSTRATION SHEET, and fill in the answers in the blank spaces, just as you did on the PRACTICE SHEET.

Do Operation A on these.			
Do Operation B on these.			
Do Operation C on these.			
Do Operation D on these.			
Do Operation E on these.			

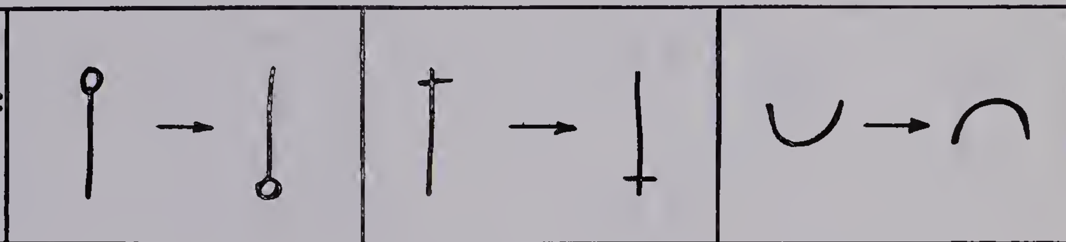
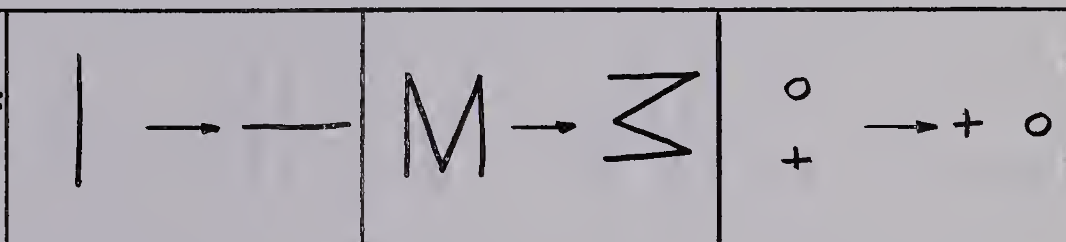
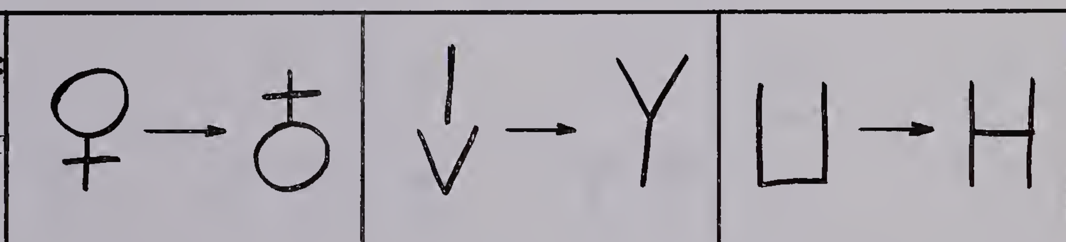
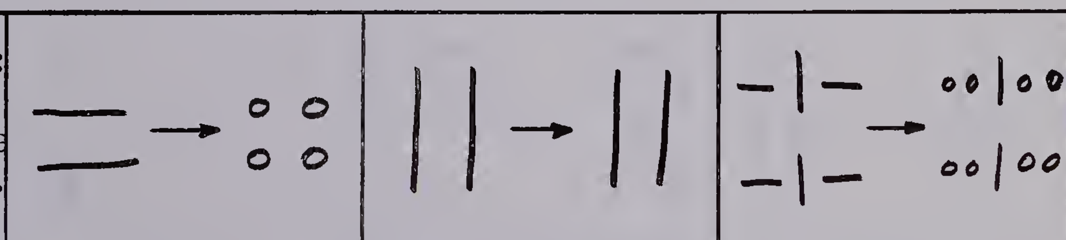
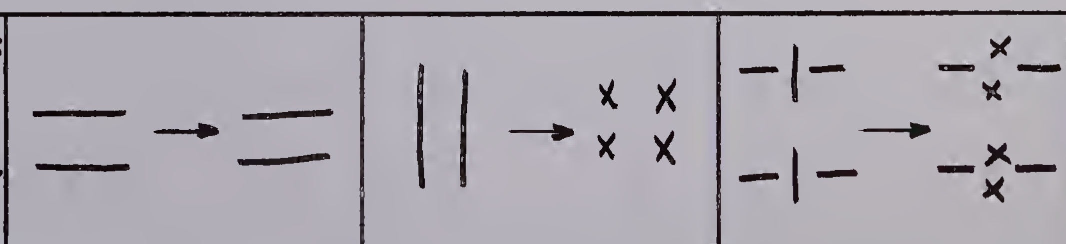
SK6: PART I

(CONTINUED)

Do Operation F on these.			
Do Operation G on these.			
Do Operation H on these.			
Do Operation I on these.			
Do Operation J on these.			

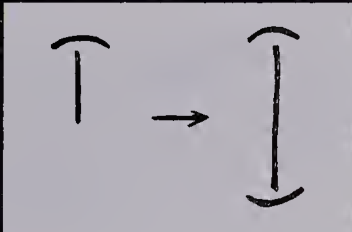
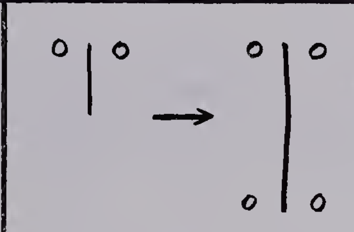
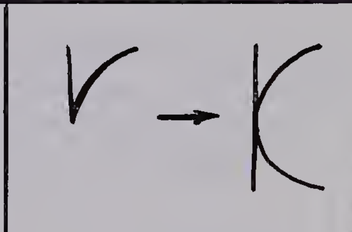
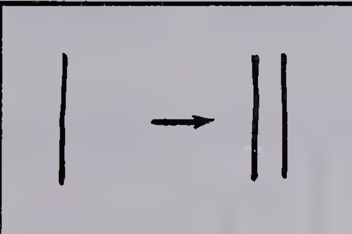
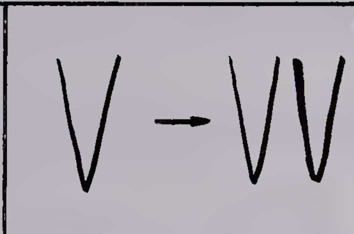
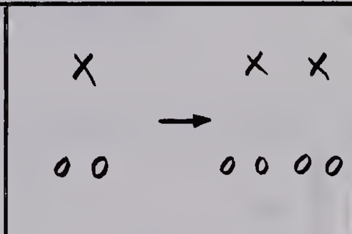
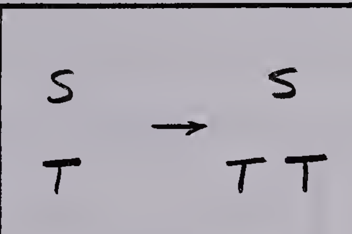
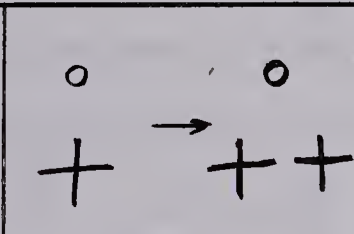
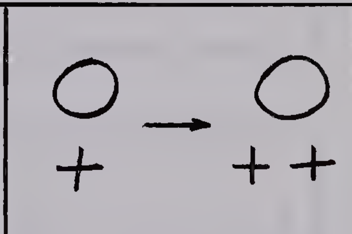
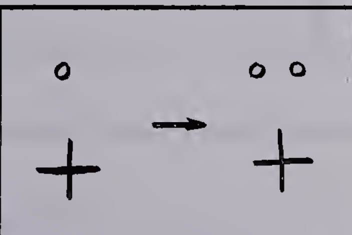
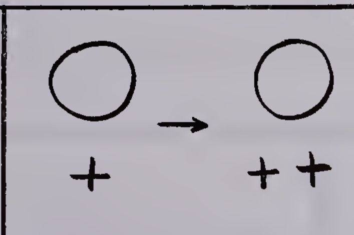
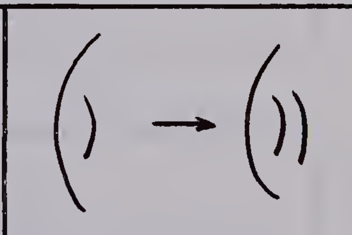
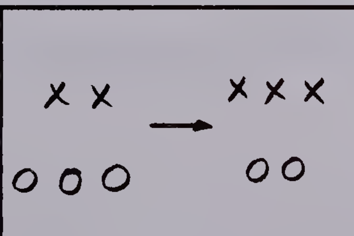
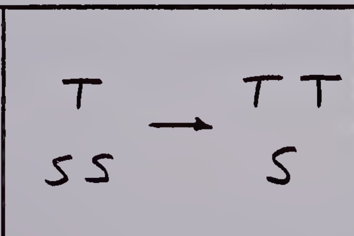
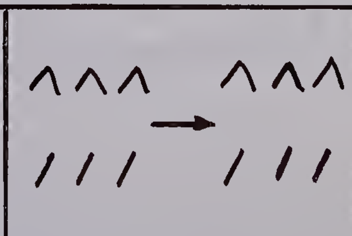
ANSWER SHEET FOR SK6, PART I

Here are the answers to the problems you did. Go through these carefully and put a tick in the right hand margin if you think that you got the whole line right. If you are not sure, ask for an explanation.

Operation A is: turn the other way up.	
Operation B is: rotate a quarter turn clockwise.	
Operation C is: interchange upper and lower parts.	
Operation D is: replace each horizontal line by two circles.	
Operation E is: replace each vertical line by two crosses.	

ANSWER SHEET FOR SK6, PART I

(CONTINUED)

Operation F is: add a symmetrical lower half.			
Operation G is: double everything.			
Operation H is: double the lower part.			
Operation I is: double the smaller part.			
Operation J is: interchange the numbers.			

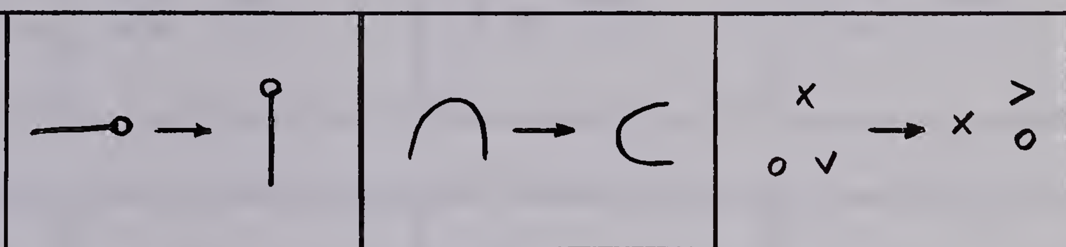
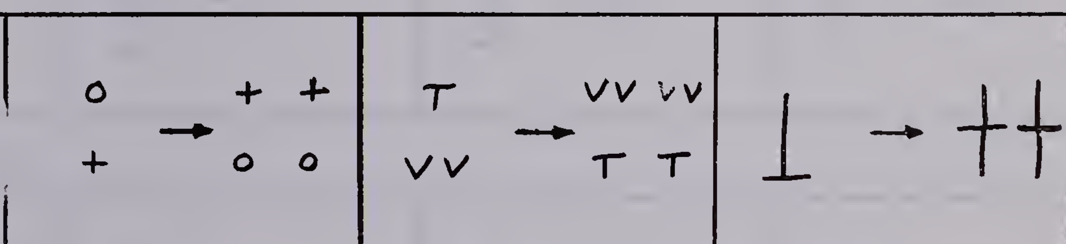
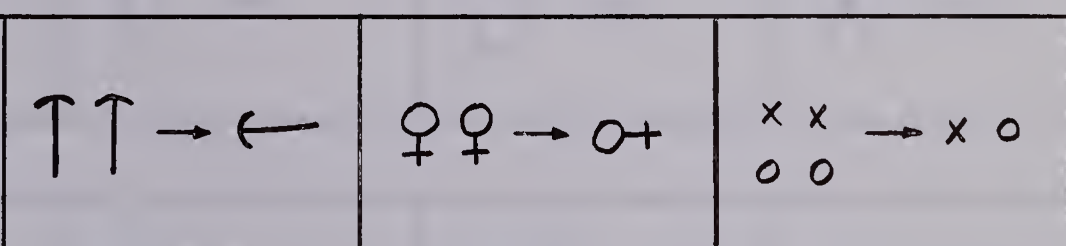
NAME .... SCHOOL ....
Last First Middle

AGE .... GRADE .... BOY GIRL DATE ....
Years (Circle One) Day Month Yr.

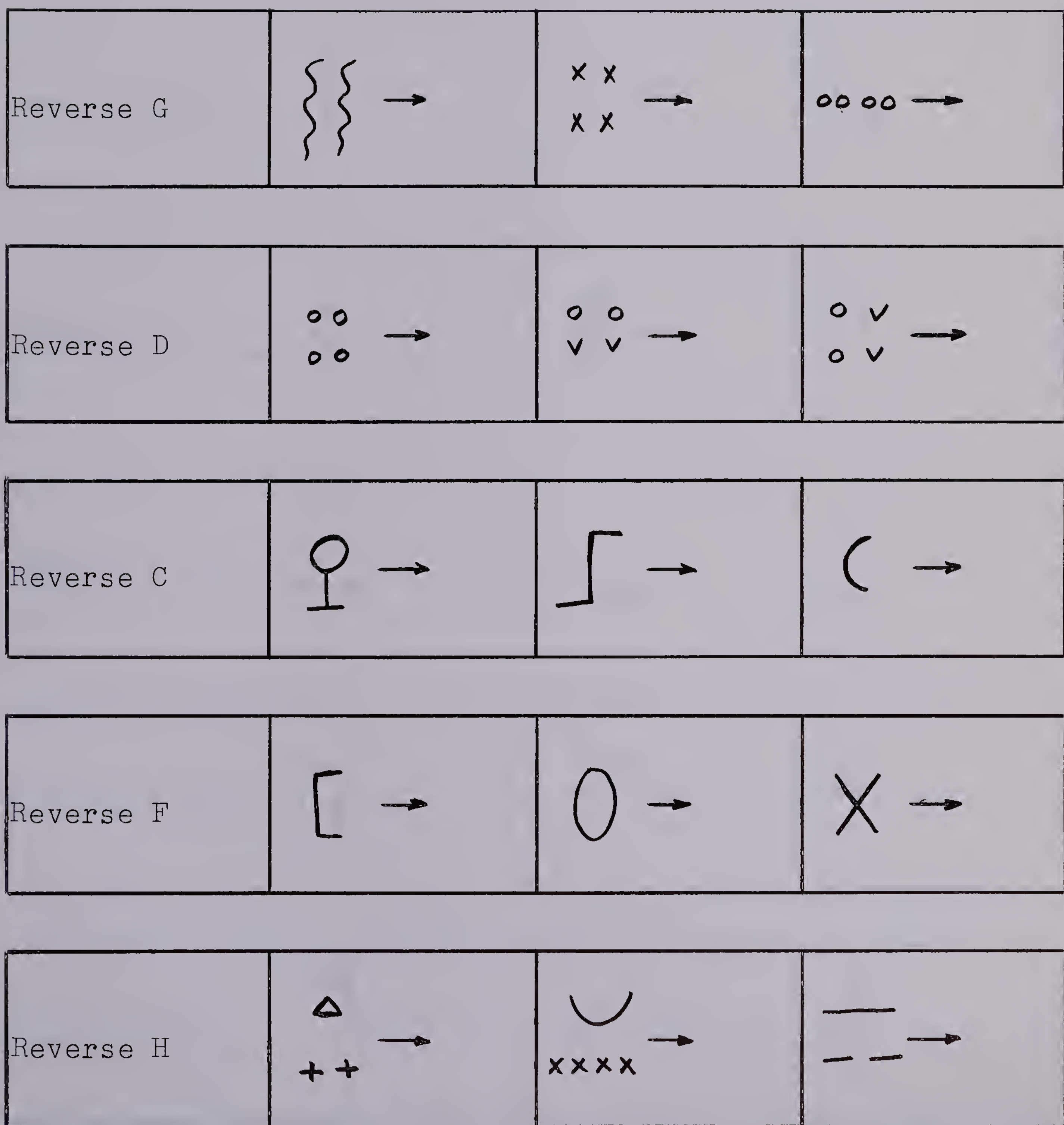
SK6: PART II

In PART II the problem is to combine the operations on the DEMONSTRATION SHEET, or to do them in reverse, or both. When combining operations, they are to be done in the order given (i.e., "Combine C and G" means "Do Operation C first and then do Operation G.")

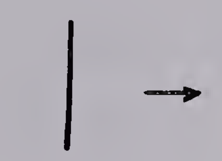
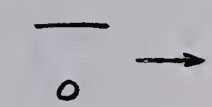
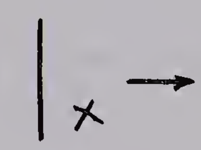
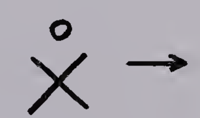
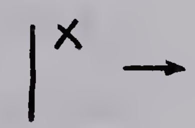
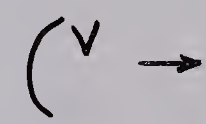
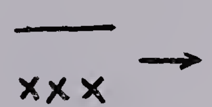
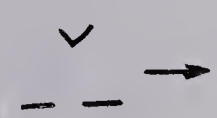
Look at the examples given below and then carry out the operations indicated on the following three pages.

EXAMPLE: Reverse B	
EXAMPLE: Combine C & G	
EXAMPLE: Reverse and Combine G & B	

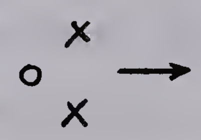
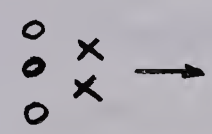
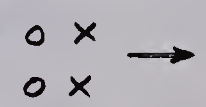
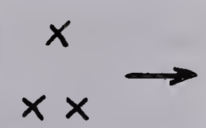
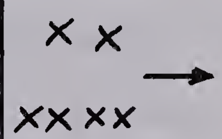
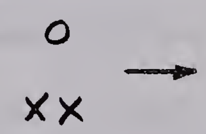
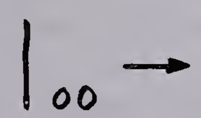
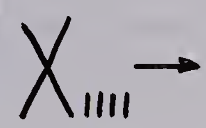
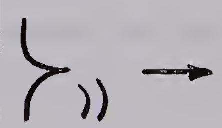
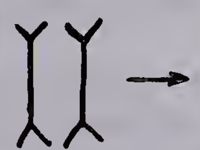
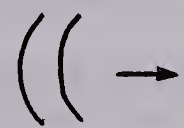
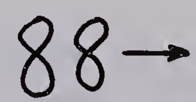
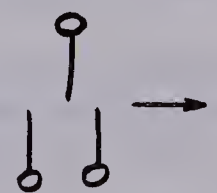
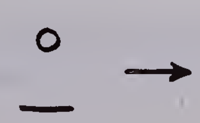
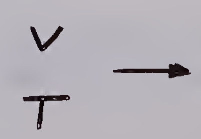
SK6: PART II



SK6: PART II

Combine E & H			
Combine A & I			
Combine D & J			
Combine B & F			
Combine F & B			

SK6: PART II

Reverse and Combine B & J			
Reverse and Combine H & E			
Reverse and Combine A & I			
Reverse and Combine F & G			
Reverse and Combine A & C			

THE TASC QUESTIONNAIRE

Permission was secured from Dr. S. B. Sarason to use items from his TASC questionnaire. The questions chosen are listed below with the slightly modified wording that was used in the present study. The original TASC questionnaire, which contains thirty items, is given on pages 87 to 89 of Sarason's book, Anxiety in Elementary School Children.

In the present study, the TASC Answer Sheet was distributed to the students immediately following administration of the DAT Space Relations test. It was explained to them that no one but the investigator would see their answers to the questions and that there were no "right" or "wrong" answers. They were asked to circle "Yes" or "No" in answer to each question to indicate how they think or feel about school. The following questions were then asked verbally by the investigator.

1. When the teacher says that she is going to call upon some students in the class to do math problems, do you hope that she will call on someone else and not on you?

2. Do you worry about being promoted, that is, passing from the eighth to the ninth grade at the end of the year?

3. Do you sometimes dream that you are in school and cannot answer the teacher's questions?

4. When the teacher is teaching math, do you feel that other students in the class understand her better than you?

5. When the teacher asks you to write on the blackboard in front of the class, does the hand you write with sometimes shake a little?

6. Do you think you worry more about school than other students?

7. If you are sick and miss school, do you worry that you will do more poorly in your schoolwork than other students when you return to school?

8. Do you sometimes dream that the teacher is angry because you do not know your lessons?

9. Are you afraid of school tests?

10. Do you worry a lot <u>before</u> you take a test?

11. Do you worry a lot <u>while</u> you are taking a test?

12. <u>After</u> you have taken a test do you worry about how well you did on the test?

13. Do you sometimes dream that you did poorly on a test you had in school?

14. When you are taking a test, does the hand you write with shake a little?

15. When the teacher says that she is going to give the class a test, do you become afraid that you will do poorly?

16. When you are taking a hard test, do you forget some things you knew very well before you started taking the test?

17. Do you often wish that you didn't worry so much about tests?

18. When the teacher says that she is going to give the class a test, do you get a nervous or funny feeling?

19. While you are taking a test do you usually think you are doing poorly?

20. While you are on your way to school, do you sometimes worry that the teacher may give the class a test?

TASC ANSWER SHEET

NAME................................ SCHOOL..................
Last First Middle

GRADE......... DATE.....................
Day Month Year

1. YES	2. YES	3. YES	4. YES	5. YES
NO	NO	NO	NO	NO
6. YES	7. YES	8. YES	9. YES	10. YES
NO	NO	NO	NO	NO
11. YES	12. YES	13. YES	14. YES	15. YES
NO	NO	NO	NO	NO
16. YES	17. YES	18. YES	19. YES	20. YES
NO	NO	NO	NO	NO
21. YES	22. YES	23. YES	24. YES	25. YES
NO	NO	NO	NO	NO

THE SMU-II TEST

The following twelve pages constitute the SMU-II test as it was used in the present study.

When the test was administered the test booklet was distributed along with the IBM answer sheet (optical scoring format), HB pencils, and a sheet of scratch paper. The instructions on the cover sheet were read to the class and thirty-five minutes of writing time was given.

SMU - II

INSTRUCTIONS

This test measures your understanding of a number of basic mathematical concepts. There are 45 multiple-choice questions in the test. Read each question carefully.

Do your rough work on the scratch paper provided.

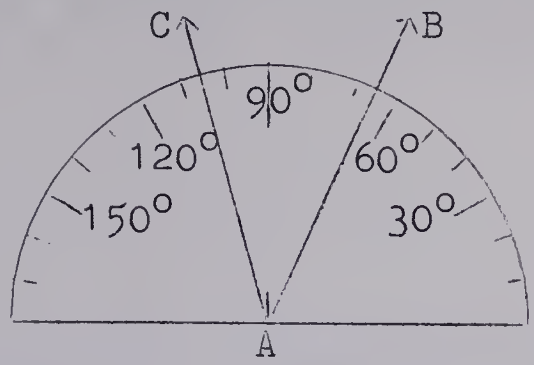
Mark your answers on the separate Answer Sheet. Mark only one answer for each question.

Fill in your NAME, SCHOOL, and the NAME OF TEST (SMU - II) in the spaces provided at the top of the Answer Sheet.

Please do not make any marks in the test booklet.

Do not turn this page until you are asked to do so.

1. The measure of angle BAC is approximately



(a) 65°
(b) 105°
(c) 40°
(d) 50°
(e) 30°

2. If a positive integer is multiplied by a positive integer, the result is

(a) always even
(b) always odd
(c) always a positive integer
(d) never a positive integer
(e) sometimes a fraction

3. $10^4 \times 10^3 =$ ________________.

(a) $10 \times 10 \times 10 \times 10 \times 10 \times 10$
(b) 10^7
(c) $10 \times 10 \times 10 \times 10^3$
(d) 10^{12}
(e) $10 \times 10 \times 10 \times 10 \times 10$

4. If the measure of one angle of a scalene triangle is 40 degrees, which of the following statements is <u>always</u> true?

(a) The measure of one of the other angles is 90 degrees.
(b) The measure of one of the other angles is 40 degrees.
(c) The sum of the measures of the other two angles is 140 degrees.
(d) Two of the sides have equal measures.
(e) The measure of one of the other angles is 140 degrees.

5. If "x" dollars is the cost per yard of a certain cloth, ten yards of cloth costs more than 12 dollars. Which statement below could be used to describe this situation?

(a) 10x is greater than 12
(b) x + 10 is greater than 12
(c) x is greater than 12
(d) 10/x is greater than 12
(e) x/10 is greater than 12

6. A design is to be made by pasting coloured 2 inch by 2 inch squares side by side to cover an 8 inch by 12 inch rectangular area. How many 2 inch by 2 inch squares would be needed?

(a) 20
(b) 24
(c) 48
(d) 16
(e) 96

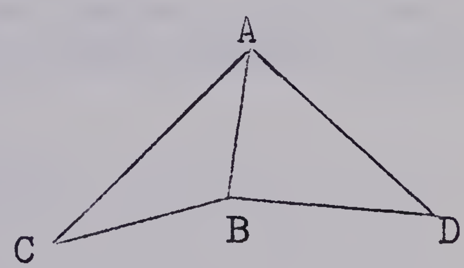
7. In which of the following is 36 completely factored (expressed as the product of primes)?

(a) 4 X 9
(b) 2 X 3 X 6
(c) 3 X 12
(d) 2 X (9 + 9)
(e) 2 X 2 X 3 X 3

8. About how many tens are there in 5472?

(a) 5.5
(b) 5,472
(c) 54.7
(d) 54,720
(e) 547

9. In the figure to the right below, line segment AB is made up of points



(a) that are on both line segment CB and line segment BD
(b) that are on both line segment AC and line segment AD
(c) that are on both triangle ABC and triangle ABD
(d) A and B
(e) that are on both line segment CA and line segment CB

10. A father is nine times as old as his son. If the son is now "n" years old, how old will the father be in 8 years?

(a) 9n + 8 years
(b) 9(n + 8) years
(c) n + 8 years
(d) n + 9(8) years
(e) 9n years

11. If both the length and the width of a rectangular solid are doubled and the height is halved, the volume of the new solid is

(a) that of the original solid
(b) one-half that of the original solid
(c) twice that of the original solid
(d) four times that of the original solid
(e) eight times that of the original solid

12. If a, b and c represent numbers, which of the following is <u>not</u> correct?

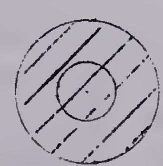
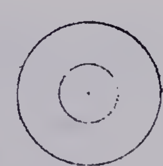
(a) $ab = ba$
(b) $a(b + c) = ab + ac$
(c) $a + (b + c) = (a + b) + c$
(d) $a + (b \times c) = (a + b) \times (a + c)$
(e) $a + b = b + a$

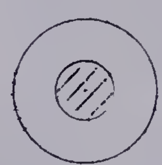
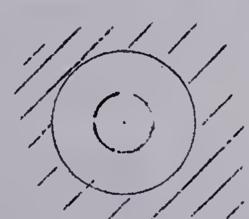
13. Which of the following represents the largest number?

(a) $3 \times 10^2 + 14 \times 10 + 5 \times 1$
(b) $4 \times 10^2 + 4 \times 10 + 5 \times 1$
(c) $4 \times 10^2 + 3 \times 10 + 15 \times 1$
(d) $2 \times 10^2 + 23 \times 10 + 5 \times 1$
(e) $3 \times 10^2 + 13 \times 10 + 25 \times 1$

14. In which of the following figures is the shaded region only that area that is in <u>both</u> the <u>exterior</u> of the smaller circle <u>and</u> the <u>interior</u> of the larger circle?

(a)

(b)

(c)


(d)

(e)


15. The current of a stream has a rate of "r" miles per hour. A motor boat goes downstream 10 miles per hour faster than the rate of the current. If the boat's speed downstream is 25 miles per hour, which mathematical statement below could be used to describe this situation?

(a) $r/10 = 2.5$
(b) $10/r = 25$
(c) $r + 10 = 25$
(d) $r - 10 = 25$
(e) $10r = 25$

16. The measure of the length of a rectangular solid is 7 feet, the measure of its width is 3 feet and the measure of its height is 4 feet. The area of its largest face would measure

(a) 21 square feet
(b) 28 square feet
(c) 49 square feet
(d) 84 square feet
(e) 12 square feet

17. If the replacement for "n" is always an odd number, then $n + 7$ is

(a) always odd
(b) always divisible by 7
(c) any one of the natural numbers (counting numbers or zero)
(d) always even
(e) always less than 8

18. $10^5 \div 10^2 =$ ________________.

(a) $10^{2\frac{1}{2}}$
(b) $10 \times 10 \times 10 \times 10$
(c) 10^7
(d) 10×10
(e) 10^3

19. Which of the following statements <u>does not</u> supply enough information for one to conclude that the geometric figures referred to are congruent.

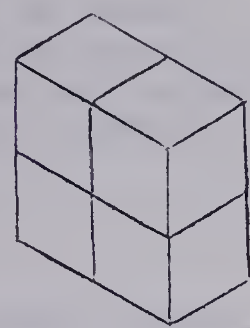
(a) The figures could be put together so that the points making up one figure would occupy exactly the same positions as the points making up the other figure.
(b) All corresponding parts of the figures have equal measures.
(c) All corresponding parts of the figures are congruent.
(d) The figures enclose exactly equal area measures.
(e) The figures have exactly the same size and shape.

20. You are "x" years old and my age is twice yours. In three years the sum of our ages will be 27 years. Which mathematical sentence below represents this situation?

(a) $2x + x = 27$
(b) $2x + x = 24$
(c) $(2x - 3) + (x + 3) = 27$
(d) $(x - 3) + (2x - 3) = 27$
(e) $(x + 3) + (2x + 3) = 27$

21. The rectangular solid to the right below is made up of a number of cubes having equal volumes. If the volume of the rectangular solid is 7 cubic units, what is the volume of each of the cubes?

(a) 28 cubic units
(b) $\frac{4}{7}$ cubic units
(c) $\frac{8}{7}$ cubic units
(d) $\frac{7}{4}$ cubic units
(e) $\frac{7}{8}$ cubic units



22. $^{+}7 + {}^{+}2({}^{-}3) =$ ______________.

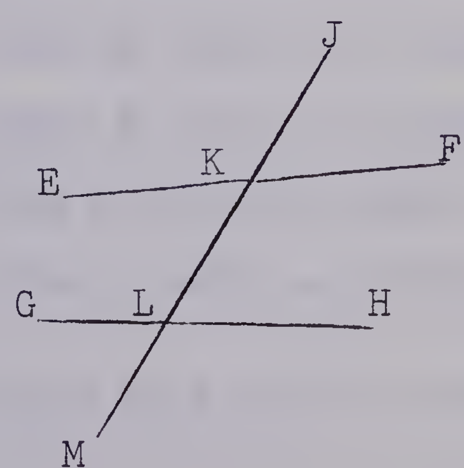
(a) $^{+}1$
(b) $^{-}27$
(c) $^{+}13$
(d) $^{+}6$
(e) $^{+}2$

23. If N represents any natural number (counting number or zero), which of the following will <u>always</u> represent an odd number?

(a) N
(b) N + 1
(c) N + 2
(d) 2N + 2
(e) 2N + 1

24. Which of the following statements about the figure to the right below is <u>false</u>?

(a) Angle EKJ is adjacent to angle JKF.
(b) Angle KLH is adjacent to angle HLM.
(c) Angle FKL is adjacent to angle JKE.
(d) Angle GLK is adjacent to angle KLH.
(e) Angle EKJ is adjacent to angle EKL.



25. Given the statement: $3a = 4b$, consider which of the following situations this statement could represent.

    (1) My salary quadrupled is matched by Jack's salary tripled.
    (2) There are four more bananas when the number of apples is increased by three.
    (3) The measures of the perimeters of an equilateral triangle and a square are equal.
    (4) Three times a certain number is equal to four times another number.

    $3a = 4b$ could represent

    (a) all four situations described above.
    (b) only (1), (3) and (4)
    (c) only (2), (3) and (4)
    (d) only (2)
    (e) only (4)

26. Twenty-seven cubes with one inch edges could be used to make

    (a) a single cube 27 inches high, 27 inches long, 27 inches wide
    (b) a single cube 9 inches high, 9 inches long, 9 inches wide
    (c) a rectangular solid 1 inch high, 25 inches long, 1 inch wide
    (d) a single cube 3 inches high, 3 inches long, 3 inches wide
    (e) a rectangular solid 10 inches high, 10 inches long, 7 inches wide

27. If the symbols @, #, $, % represent numbers arranged in order from smallest to largest, which of the following statements is true?

    (a) $ represents a number larger than @ does but smaller than # does.
    (b) # represents a number smaller than % does but larger than @ does.
    (c) @ represents a number smaller than % does but larger than # does.
    (d) % represents a number larger than $ does but smaller than # does.
    (e) @ represents a number smaller than # does but larger than $ does.

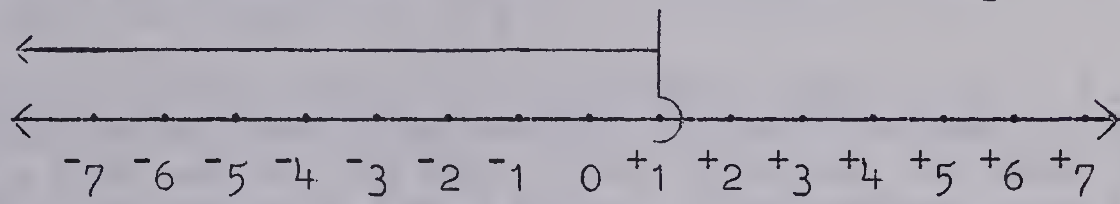
28. Which of the following procedures would give you the correct product for 356×742?

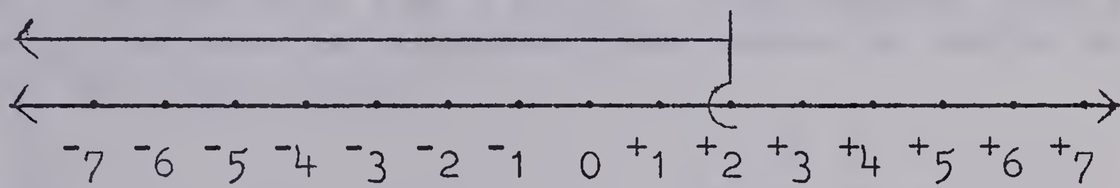
    (a) Add the products: 356×2; 356×4; 356×7
    (b) Add the products: 356×7; 356×40; 356×200
    (c) Add the products: 742×6; 742×5; 742×3
    (d) Add the products: 742×6; 742×56; 742×300
    (e) Add the products: 742×300; 742×50; 742×6

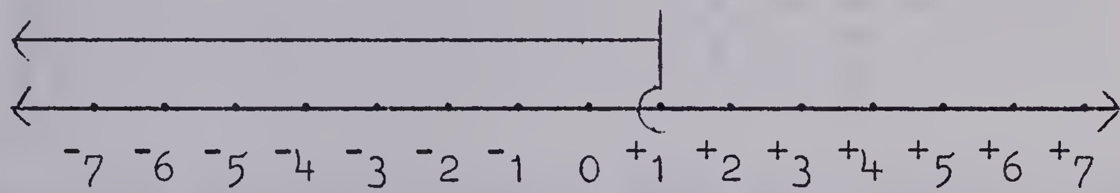
29. If the radius of a circle is doubled, the <u>area</u> enclosed by the newly formed circle is

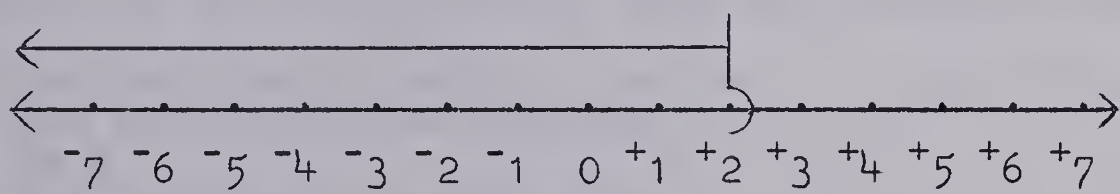
(a) equal to that enclosed by the original circle
(b) one-half that enclosed by the original circle
(c) twice that enclosed by the original circle
(d) four times that enclosed by the original circle
(e) eight times that enclosed by the original circle

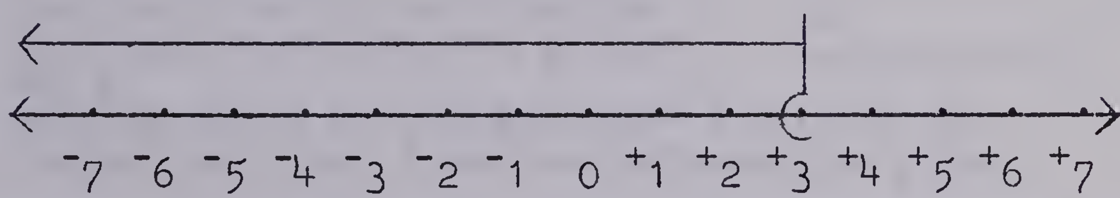
30. Which of the diagrams below best illustrates the set of real numbers that could be used as replacements for "x" to make a true statement of "2x + 1 is less than 3"?

(a) 

(b) 

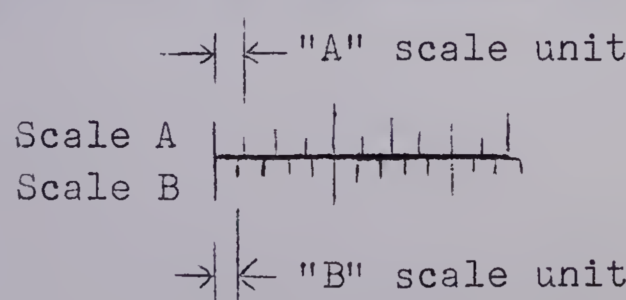
(c) 

(d) 

(e) 

31. Suppose that a line segment is measured with Scale A and then with Scale B. The ratio of the number of "A" scale units to the number of "B" scale units that would be used in making the measurements is

(a) 3/4
(b) 5/4
(c) 2/5
(d) 4/5
(e) 5/6



32. If a, b and c represent numbers, which of the following statements is <u>false</u>?

(a) For any two specific numbers, a and b, only one of the following can be true: a is less than b, a = b, a is greater than b.
(b) If a is less than b and b is less than c, then a must be less than c.
(c) If a is less than b and b is greater than c, then a must be less than c.
(d) If a is less than b, then a + c must be less than b + c.
(e) If a is less than b and c is greater than 0, then a $\times$ c must be less than b $\times$ c.

33. Imagine a place where the inhabitants have only 6 fingers altogether so that they use six in their system of numeration just as we use ten in ours. For instance, to them "23" means 2 groups of 6 and 3 units. What numeral would be used in such a system to represent the number of marks in the set to the right below?

(a) 33
(b) 23
(c) 21
(d) 43
(e) 29

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* * * * * *
* * * * * *
* * * * * *
* * *
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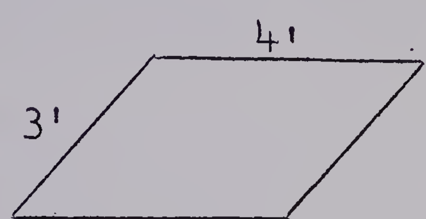
34. The area of the parallelogram shown at the right below



(a) can be found by adding 3 and 4
(b) can be found by multiplying 3 by 4
(c) can be found by multiplying 3 by 4 and dividing the product by 2
(d) can be found by multiplying (3 + 4) by $\frac{1}{2}$
(e) cannot be found from the information given

35. If a speed of 38 miles per hour is equivalent to a speed of 33 knots, which of the following statements would enable one to convert a speed in miles per hour to knots ("m" represents the number of miles per hour, and "k" represents the equivalent number of knots)?

(a) $38m = 33k$
(b) $\frac{k}{m} = \frac{38}{33}$
(c) $\frac{m}{k} = \frac{38}{33}$
(d) $\frac{38}{k} = \frac{33}{m}$
(e) $m - 5 = k$

36. The measures of the perimeters of a square, a rectangle and an equilateral triangle are equal. Which of the following statements about them is true?

(a) The area enclosed by the square is greater than that enclosed by either of the other two figures.
(b) The square and the rectangle enclose equal areas.
(c) The square and the triangle enclose equal areas.
(d) The area enclosed by the triangle is one-half that enclosed by the rectangle.
(e) The area enclosed by the rectangle is greater than that enclosed by either of the other two figures.

37. Consider the following:

$$0.73 + 0.84 = \left(\frac{1}{100} \times 73\right) + \left(\frac{1}{100} \times 84\right)$$
$$= \frac{1}{100}(73 + 84)$$

Which of the following properties of a number system has been used in the steps shown above?

(a) $a + b = b + a$
(b) $ab = ba$
(c) $a + (b + c) = (a + b) + c$
(d) $a(bc) = (ab)c$
(e) $ab + ac = a(b + c)$

38. Consider a numeration system in which only the following symbols are used: $ is 0, % is 1, ¢ is 2, & is 3. Which of the choices below would be the next three numerals in the sequence:

%, ¢, &, %$, %%, %¢, %&, ¢$, . . .

(a) ¢¢, ¢&, &$
(b) ¢%, ¢¢, ¢&
(c) &$, &%, &¢
(d) ¢&, &$, &%
(e) &%, &¢, &&

39. Which of the following statements is <u>false</u>?

(a) A circle with measurable radius can be drawn through any three points.
(b) A line segment can intersect a circle at one point, two points, or not at all.
(c) A line segment drawn through the center of a circle and having its end points on the circle is called a diameter of the circle.
(d) All of the points that make up a circle are at equal distances from the center of the circle.
(e) The interior of a circle does not include the circle.

40. In my pocket I have $3.90. I have "n" nickels. I have three times as many dimes as nickels, and I have six more quarters than nickels. Which of the following mathematical statements could be used to find out how many nickels I have?

(a) $n + 3n + n + 6 = 390$
(b) $5n + 6 = 390$
(c) $5n + 10(3n) + 25(n + 6) = 390$
(d) $5n + 10n + 25n + 6 = 390$
(e) $5n + 10(3n) + 25(6n) = 390$

41. Suppose that a sculptor wants to make a scale model of a monument by reducing each measurement to 1/3 that of the original. The ratio of the volume of the material that would be required for the scale model to that required for the monument would be

(a) 1/3
(b) 1/27
(c) 9/1
(d) 1/9
(e) 3/1

42. Which of the following means the same thing as $x = a \div b$, in which a, b and x represent numbers other than zero?

(a) $b \times x = a$
(b) $a \times x = b$
(c) $x \div b = a$
(d) $x = b \div a$
(e) $b \div x = a$

43. In the numeral 632, the 6 represents a value that is

(a) twice the value represented by the 3
(b) three times the value represented by the 2
(c) twenty times the value represented by the 3
(d) thirty times the value represented by the 2
(e) two hundred times the value represented by the 3

44. Mr. Jones' present income is "m" dollars per month. An increase of 20% would give him an income of $300.00 per month. Which of the following statements would enable you to find his correct present monthly income?

(a) $\frac{m}{300} = \frac{20}{100}$

(b) $\frac{m}{300} = \frac{100}{120}$

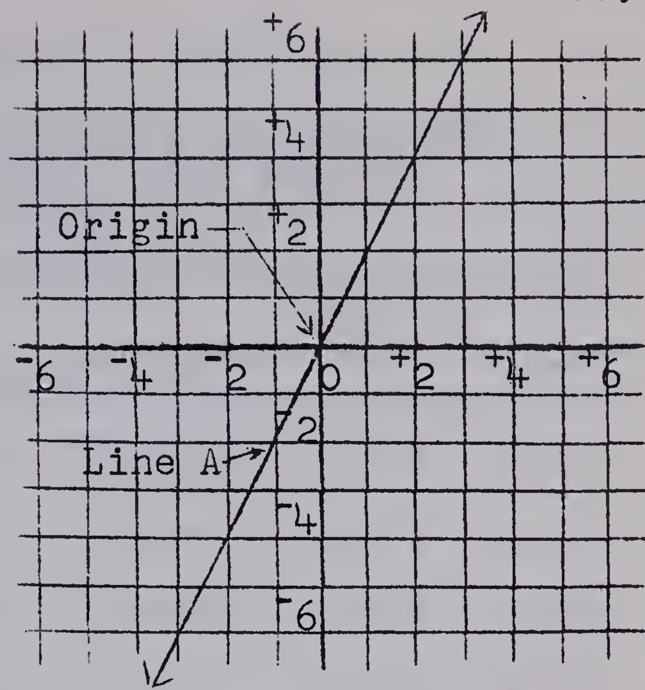
(c) $\frac{300 - m}{300} = \frac{20}{100}$

(d) $\frac{m}{300} = \frac{120}{100}$

(e) $m + 20 = 300$

11.

45. Referring to the diagram to the right and counting "a" units horizontally from the origin and then "b" units vertically, for which of the following number replacements for "a" and "b" would you locate a point on line A?



(a) $a = {}^{+}2$, $b = {}^{+}3$

(b) $a = {}^{-}2$, $b = {}^{+}4$

(c) $a = {}^{+}3$, $b = {}^{-}6$

(d) $a = {}^{+}1$, $b = {}^{+}2$

(e) $a = {}^{-}1$, $b = {}^{+}2$

TABLE XXIV

CORRELATIONS, MEANS, AND STANDARD DEVIATIONS FOR THE SUM AND PRODUCT VARIABLE SCORES REFERRED TO IN THE PRESENT REPORT (N=131)*

	VR+NA	SK6(1)+ SK6(2)	VR×NA	AR×SK6(1)	SK4(1)× TASC	SMU-II
VR	0.886	0.506	0.854	0.550	0.019	0.537
NA	0.829	0.530	0.837	0.499	0.023	0.568
AR	0.551	0.617	0.537	0.870	0.065	0.482
SR	0.419	0.553	0.408	0.551	0.027	0.334
SK4(1)	0.391	0.482	0.365	0.471	0.598	0.362
SK4(2)	0.525	0.500	0.528	0.518	0.295	0.394
SK6(1)	0.505	0.855	0.507	0.884	0.137	0.478
SK6(2)	0.565	0.929	0.560	0.643	0.176	0.491
TASC	−0.347	−0.226	−0.349	−0.294	0.635	−0.196
VR+NA	1.000	0.601	0.984	0.613	———	0.640
SK6(1)+ SK6(2)	0.601	1.000	0.599	0.827	———	0.541
VR×NA	0.984	0.599	1.000	0.615	−0.001	0.654
AR×SK6(1)	0.613	0.827	0.615	1.000	0.092	0.562
SK4(1)× TASC	———	———	−0.001	0.092	1.000	0.113
SMU-II	0.640	0.541	0.654	0.562	0.113	1.000
MEAN	47.31	33.96	580.90	675.56	102.00	14.14
STD. DEV.	11.74	11.78	293.66	283.07	56.99	51.30

*See Table VII, page 216, for the rest of the intercorrelations. The sum and product variables are referred to in Tables X, XI, and XII and the text surrounding those tables.

FINDINGS FROM A FOLLOW-UP INVESTIGATION

In August, 1967, grade nine departmental mathematics examination scores were secured for 125 of the students who had participated in the grade eight portion of the study. Stepwise regression analyses were then carried out as in the original study but with the grade nine mathematics score as the criterion.

Displayed in Table XXV is the matrix of Pearson product-moment correlations among the scores obtained by the 125 students on the VR, NA, AR, SR, SK4(1), SK4(2), SK6(1), SK6(2), TASC, VR+NA, SK6(1)+SK6(2), SMU-II, and grade nine mathematics (GR IX) tests. All of the correlations with the grade nine mathematics criterion were significantly different from zero (0.01 level test). In addition to the correlation matrix, Table XXV contains the means and standard deviations of the thirteen variables. It is interesting to note that each of the predictor variables had a higher correlation with the grade nine mathematics variable than it did with SMU-II, even though a year had elapsed before the grade nine examinations were written. It is not surprising that the single variable having the greatest correlation with GR IX mathematics was SMU-II, the grade eight mathematics criterion.

Table XXVI summarizes the findings from a stepwise

[illegible]

[illegible]

TABLE XXV

CORRELATIONS, MEANS, AND STANDARD DEVIATIONS FOR THE SCORES OF 125 GRADE EIGHT STUDENTS FROM A BATTERY OF ABILITY, REFLECTIVE INTELLIGENCE, MATHEMATICS UNDERSTANDING AND NINTH GRADE MATHEMATICS TESTS*

	VR	NA	AR	SR	SK4 (1)	SK4 (2)	SK6 (1)	SK6 (2)	TASC	VR+NA	SK6 (1+2)	SMU-II	GR IX
VR	1.000	0.461	0.469	0.378	0.315	0.459	0.479	0.432	−0.292	0.882	0.503	0.529	0.586
NA		1.000	0.474	0.395	0.314	0.414	0.387	0.539	−0.303	0.824	0.529	0.572	0.672
AR			1.000	0.485	0.352	0.445	0.585	0.526	−0.231	0.551	0.613	0.489	0.615
SR				1.000	0.286	0.382	0.519	0.487	−0.234	0.451	0.558	0.367	0.496
SK4(1)					1.000	0.486	0.486	0.381	−0.155	0.367	0.473	0.343	0.426
SK4(2)						1.000	0.469	0.463	−0.100	0.512	0.519	0.406	0.448
SK6(1)							1.000	0.602	−0.248	0.511	0.856	0.476	0.557
SK6(2)								1.000	−0.138	0.561	0.928	0.488	0.586
TASC									1.000	−0.347	−0.205	−0.173	−0.284
VR+NA										1.000	0.602	0.641	0.730
SK6(1+2)											1.000	0.539	0.640
SMU-II												1.000	0.681
GR IX													1.000
MEAN	23.97	23.01	33.16	27.30	10.24	15.00	19.44	14.16	10.23	46.98	33.60	14.09	53.32
STD. DEV.	7.42	6.17	7.69	9.50	3.73	4.47	5.53	7.67	4.03	11.63	11.85	5.17	21.10

*The ninth grade test was written a year after the other tests were written.

TABLE XXVI

SIGNIFICANT PREDICTORS OF GRADE IX DEPARTMENTAL MATHEMATICS SCORES IN THE ORDER ENTERED DURING STEPWISE REGRESSION ANALYSIS

Step, n	Predictors Included in Regression Equation	Multiple Correlation, R	F_R	P_R	Percentage of Variance Accounted for	t_n
1	NA	0.6728	101.74	0.000	45.27	———
2	NA, AR	0.7522	79.50	0.000	56.58	5.64
3	NA, AR, VR	0.7821	63.54	0.000	61.17	3.78
4	NA, AR, VR, SK6(1)	0.7925	50.65	0.000	62.80	2.29
Best Prediction Equation: $\hat{X}_{GRIX} = {}^{-}18.88 + 1.41X_{NA} + 0.82X_{AR} + 0.72X_{VR}$						
Coefficient *t* ratios:		8.82	6.58	5.63		

$t_{.01}\ (121 \pm 1df) = 2.36$

regression analysis in which NA, VR, AR, SR, SK4(1), SK4(2), SK6(1), SK6(2), and TASC were potential predictors and GR IX mathematics score was the criterion. As can be seen from the table, the first variable entered into the regression equation was NA, followed by AR, VR, and SK6(1). However, the SK6(1) variable did not make a significant 0.01 level contribution (Had a 0.05 level test been used SK6(1) would have been included as a significant predictor.). The multiple correlation between the SMU-II scores and the composite NA, AR, and VR scores was 0.7821, accounting for 61.17 per cent of the criterion variance.

Table XXVII summarizes the findings from a stepwise regression analysis in which VR+NA and SK6(1)+SK6(2) were included as potential predictor variables. The multiple correlation coefficient obtained from the regression equation using VR+NA, AR, and SK6(1)+SK6(2) as predictors was 0.7896, accounting for 62.35 per cent of the GR IX criterion variance.

The best weighted sum of two of Skemp's test scores for predicting GR IX scores was $1.21X_{SK6(1)} + 1.08X_{SK6(2)}$, which produced a multiple correlation of 0.6398. As with the grade eight criterion, the SK6(1)+SK6(2) combination (correlation with GR IX of 0.6396) is suggested as a useful score. In fact, disregarding the SMU-II score, the

TABLE XXVII

PREDICTION OF GRADE NINE DEPARTMENTAL MATHEMATICS SCORES USING VR+NA AND SK6(1)+SK6(2) VARIABLES

Step, n	Predictors Included in Regression Equation	Multiple Correlation, R	F_R	P_R	Percentage of Variance Accounted for	t_n
1	VR+NA	0.7304	140.67	0.000	53.35	——
2	VR+NA, AR	0.7737	91.02	0.000	59.87	4.53
3	VR+NA, AR, SK6(1)+SK6(2)	0.7896	66.79	0.000	62.35	2.82
4	VR+NA, AR, SK6(1)+SK6(2), SR	0.7937	51.06	0.000	62.99	1.44

Best Prediction Equation: $\hat{X}_{GR\ IX} = {}^{-}15.41 + 0.87X_{VR+NA} + 0.60X_{AR} + 0.39X_{SK6(1)+SK6(2)}$

Coefficient *t* ratios: 6.59 2.99 2.82

$t_{.01}\ (121 \pm 1df) = 2.36$

best scores from 40-minute testing sessions for predicting a student's grade nine mathematics score were NA and SK6(1)+SK6(2).

Using only VR+NA and SK6(1)+SK6(2) as predictors, an optimum weighting of $0.98X_{VR+NA} + 0.56X_{SK6(1)+SK6(2)}$ produced a multiple correlation of 0.7722 with the GR IX criterion. As with the SMU-II criterion, a $2X_{VR+NA} + X_{SK6(1)+SK6(2)}$ weighting is suggested. Such a combination of scores produced a correlation of 0.7719 with the GR IX criterion, a correlation not practically different from any of the correlations produced from optimally weighted sums of any of the grade eight predictor scores.

RAW SCORES

On the following four pages are listed the raw scores obtained by the students involved in the grade eight portion of the study in May and June, 1966. On the ten pages then following are listed the raw scores obtained by the ten to sixteen-year-olds who were involved in the longitudinal study.

The symbols used as column headings are to be interpreted as follows:

ID: Identification Number

VR: DAT Verbal Reasoning Score

NA: DAT Numerical Ability Score

AR: DAT Abstract Reasoning Score

SR: DAT Space Relations Score

SK4(1): Skemp's Concept Formation Score

SK4(2): Skemp's Reflective Action with Concepts Score

SK6(1): Skemp's Operations Formation Score

SK6(2): Skemp's Reflective Action with Operations Score

TASC: Sarason's Test Anxiety Scale for Children Score

SMU-II: Special Mathematical Understandings Score

GR IX: Grade Nine Departmental Mathematics Examination Score

In the column headed SEX, boys are coded 0 and girls are coded 1.

ID	AGE	SEX	VR	NA	AR	SR	SK4-1	SK4-2	SK6-1	SK6-2	TASC	SMUII	GR IX
1	13	0	35	35	45	40	10	19	26	22	6	23	83
2	14	0	27	26	29	27	6	14	25	12	5	19	38
3	14	0	39	21	37	31	12	7	20	4	3	10	69
4	14	0	17	16	29	11	5	10	16	12	9	7	57
5	14	0	33	32	40	32	13	21	27	26	6	18	76
6	13	1	23	18	31	18	12	16	18	5	12	15	50
7	11	0	29	32	43	35	15	22	28	20	6	19	98
8	13	1	23	24	27	15	10	11	18	8	10	9	61
9	14	1	26	18	24	25	10	13	14	7	13	16	76
10	13	0	28	32	43	44	13	19	26	28	8	24	101
11	14	0	22	36	44	32	13	17	24	19	12	21	68
12	13	0	29	26	38	32	7	15	22	22	8	19	84
13	14	0	12	24	38	29	14	14	21	8	10	18	76
14	13	1	26	28	42	30	15	14	26	28	2	19	87
15	13	1	29	28	33	38	14	20	19	12	18	19	54
16	15	0	13	28	43	41	15	9	17	13	11	11	66
17	14	0	26	31	44	46	12	20	25	31	7	17	91
18	13	1	26	28	29	10	10	18	17	16	12	13	50
19	13	0	32	28	40	41	14	20	28	16	3	16	67
20	14	0	25	19	30	16	14	14	18	7	12	6	41
21	13	0	26	26	39	35	13	19	22	23	11	13	81
22	13	0	27	20	38	26	15	21	23	25	20	13	64
23	13	1	41	39	41	36	13	21	29	17	1	25	114
24	13	0	37	33	40	42	13	22	25	26	2	26	105
25	14	1	40	33	43	37	14	27	25	29	9	18	83
26	13	0	25	23	32	18	9	13	14	8	7	15	58
27	14	0	10	18	36	29	3	14	14	6	10	10	45
28	14	0	25	32	40	21	15	18	24	11	0	18	75
29	14	0	18	20	34	22	11	11	16	10	13	2	45
30	13	1	16	23	29	24	13	16	19	21	17	12	36
31	14	0	31	32	40	31	14	17	20	22	17	16	78
32	13	1	24	23	37	40	12	16	23	31	9	14	60
33	14	1	27	19	41	26	12	21	22	19	4	10	50
34	14	0	22	30	38	36	2	11	19	23	10	13	51

ID	AGE	SEX	VR	NA	AR	SR	SK4-1	SK4-2	SK6-1	SK6-2	TASC	SMUII	GR IX
35	13	1	24	25	32	39	4	15	7	7	10	9	49
36	13	0	29	22	29	35	6	8	24	12	11	10	52
37	13	1	34	31	34	30	12	24	28	29	10	17	81
38	13	1	19	33	34	24	12	25	18	16	11	13	97
39	12	0	14	22	29	36	14	20	25	20	9	7	64
40	14	0	22	13	37	27	12	8	14	11	13	15	61
41	13	0	31	29	46	51	12	21	26	16	11	21	71
42	13	1	6	18	26	34	5	8	16	10	16	13	47
43	15	0	13	20	32	26	7	15	10	7	12	9	51
44	13	1	16	21	8	35	14	17	23	13	14	10	53
45	13	1	31	26	42	27	14	10	25	17	14	19	75
46	13	0	18	22	31	31	6	16	14	15	15	13	53
47	14	0	23	24	38	32	14	17	16	16	11	11	74
48	13	1	17	17	35	27	9	20	21	15	9	13	37
49	13	1	39	21	41	46	12	21	24	14	3	23	90
50	13	1	28	22	34	30	14	22	26	27	13	12	77
51	13	1	22	15	33	34	14	12	25	7	7	11	63
52	13	0	25	26	25	23	9	8	10	7	9	12	67
53	14	1	19	30	31	41	9	10	19	21	7	11	35
54	13	1	25	21	33	42	8	17	21	20	10	15	57
55	13	0	28	16	33	34	15	8	22	28	10	12	64
56	13	0	28	20	36	35	15	17	28	10	10	16	67
57	13	0	39	27	36	44	15	17	25	9	8	17	93
58	14	0	21	24	32	30	12	13	19	27	7	7	36
59	14	1	23	21	35	27	10	15	22	21	14	17	68
60	13	0	36	34	44	41	10	19	25	33	11	24	116
61	14	0	21	23	39	47	10	17	20	14	6	16	60
62	13	0	18	18	38	20	13	9	22	14	6	15	46
63	13	1	20	24	25	23	12	9	17	16	7	10	67
64	14	1	21	22	32	30	4	11	18	8	5	10	75
65	13	1	21	22	26	17	2	5	18	10	8	15	49
66	13	0	24	28	35	24	14	16	21	18	5	26	86
67	16	0	28	12	31	29	12	16	20	17	8	15	25
68	13	1	34	37	42	33	13	18	27	19	12	22	95

ID	AGE	SEX	VR	NA	AR	SR	SK4-1	SK4-2	SK6-1	SK6-2	TASC	SMUII	GR IX
69	13	1	22	18	42	18	14	21	23	12	10	13	49
70	13	1	33	18	37	29	4	14	22	2	16	4	41
71	13	1	28	21	36	33	5	12	18	14	15	14	65
72	13	1	28	25	32	14	7	18	15	10	6	7	35
73	13	1	23	24	39	24	15	11	17	10	10	13	73
74	13	1	30	26	37	28	5	14	23	13	7	20	78
75	13	1	20	18	30	26	7	11	13	16	7	10	42
76	14	0	13	13	19	30	11	19	22	3	12	9	21
77	14	1	21	17	32	11	12	15	16	3	9	15	49
78	13	0	28	26	31	36	9	16	20	18	8	8	56
79	13	1	30	26	30	25	14	18	16	10	9	13	59
80	13	1	20	20	22	12	3	13	13	14	15	10	47
81	14	0	16	16	38	28	10	13	22	5	7	8	30
82	13	1	24	19	30	20	3	11	12	8	18	16	61
83	13	1	15	14	20	18	7	9	15	9	12	10	30
84	13	0	38	28	37	27	13	18	21	22	11	26	80
85	13	0	39	9	28	25	13	17	15	11	9	12	25
86	13	1	25	22	2	28	12	11	21	11	11	13	48
87	14	0	31	21	32	22	14	20	20	6	6	11	48
88	13	1	24	19	41	20	7	13	22	18	12	16	68
89	14	1	23	15	33	15	11	15	12	3	13	9	41
90	13	1	28	13	39	31	9	19	26	19	16	13	55
91	14	1	14	10	22	14	7	12	19	4	12	8	29
92	13	1	20	22	41	12	11	14	23	18	12	12	66
93	13	0	39	34	45	33	11	17	26	32	3	19	79
94	13	0	26	18	30	21	6	11	20	12	8	12	66
95	13	1	27	20	35	23	4	14	21	14	15	15	42
96	12	1	25	24	42	28	14	15	23	21	17	19	52
97	14	0	14	13	11	12	4	10	3	1	10	6	16
98	14	0	24	17	27	17	5	9	16	3	10	6	20
99	14	0	19	19	40	42	14	25	28	15	8	14	52
100	13	0	31	28	33	25	9	20	11	19	9	25	88
101	15	0	19	18	27	21	12	13	22	18	14	14	36
102	14	1	11	17	29	17	10	12	17	10	12	13	43

ID	AGE	SEX	VR	NA	AR	SR	SK4-1	SK4-2	SK6-1	SK6-2	TASC	SMUII	GR IX
103	13	0	32	30	36	13	8	15	23	17	5	19	62
104	13	1	36	26	42	37	14	23	27	25	15	25	90
105	13	1	15	20	37	20	5	13	19	7	18	10	37
106	13	1	21	13	35	26	6	16	19	10	16	9	54
107	14	0	29	26	40	38	15	20	20	15	18	25	70
108	14	1	29	31	38	28	13	14	21	14	6	21	70
109	14	1	20	14	19	18	4	10	4	3	10	8	25
110	13	1	23	27	30	20	6	12	19	2	12	10	42
111	13	1	19	26	26	11	12	19	7	4	12	11	33
112	14	1	15	22	19	12	3	6	11	5	16	15	27
113	15	0	15	25	31	39	6	11	16	11	3	12	34
114	13	1	20	21	33	27	9	14	20	14	13	13	42
115	14	0	15	14	22	20	6	8	11	0	10	11	40
116	13	1	18	15	24	15	6	14	8	3	13	8	19
117	13	0	15	24	31	25	10	14	17	6	11	13	43
118	14	1	15	28	32	12	12	15	17	18	11	9	60
119	14	1	15	22	16	12	12	12	6	4	13	10	27
120	16	1	17	24	27	22	7	13	5	5	12	14	38
121	15	0	14	18	24	20	5	10	18	12	16	11	32
122	13	1	22	30	28	26	14	16	18	18	11	14	51
123	14	0	11	18	33	19	11	9	23	12	10	19	56
124	13	0	22	25	35	37	13	12	20	15	9	9	56
125	13	0	24	22	27	11	14	15	22	15	15	22	53
126	16	1	29	29	43	54	13	22	22	24	11	12	-1
127	13	0	23	31	35	23	10	14	24	20	7	15	-1
128	14	1	18	16	34	38	3	9	17	11	15	8	-1
129	13	0	27	20	36	42	13	8	24	26	3	19	-1
130	13	1	34	25	37	28	13	18	24	16	10	18	-1
131	14	1	40	33	38	24	14	20	19	22	9	19	-1

ID	AGE	SEX	SK4-1	SK4-2	SK6-1	SK6-2
1	10	1	9	15	20	10
2	10	1	1	7	6	7
3	10	1	5	9	19	17
4	10	1	6	5	6	9
5	10	1	7	14	18	7
6	10	0	7	9	6	3
7	10	1	12	11	11	7
8	10	0	10	11	10	3
9	10	0	14	16	21	16
10	10	0	9	12	5	11
11	10	0	6	16	10	12
12	10	1	13	17	15	12
13	10	1	12	12	18	7
14	10	0	12	18	21	2
15	10	1	4	9	19	5
16	10	1	1	3	19	14
17	10	1	7	10	18	6
18	10	1	7	3	22	7
19	10	0	12	10	17	8
20	10	0	11	13	25	28
21	10	1	15	18	25	25
22	10	1	1	3	16	5
23	10	1	14	18	10	0
24	10	1	10	14	15	5
25	10	0	13	18	22	22
26	10	1	6	13	23	8
27	10	0	10	9	20	22
28	10	1	7	13	20	15
29	10	1	13	18	14	16
30	10	0	11	10	23	22
31	11	0	12	15	13	14
32	11	0	7	9	12	1
33	11	0	6	11	7	1
34	11	0	3	6	3	1

ID	AGE	SEX	SK4-1	SK4-2	SK6-1	SK6-2
35	11	0	9	17	17	15
36	11	1	5	10	17	10
37	11	1	10	13	7	6
38	11	0	6	13	22	28
39	11	0	10	8	8	9
40	11	0	11	16	19	12
41	11	1	14	18	27	12
42	11	0	12	15	15	10
43	11	0	11	19	20	7
44	11	1	14	14	15	19
45	11	0	6	13	17	6
46	11	0	5	8	16	0
47	11	1	9	10	15	8
48	11	0	13	17	22	10
49	11	0	8	11	16	14
50	11	1	12	13	17	10
51	11	1	11	19	9	5
52	11	1	8	13	17	5
53	11	1	15	18	25	15
54	11	0	13	26	26	13
55	11	0	14	13	28	26
56	11	1	15	28	24	20
57	11	1	1	3	17	4
58	11	1	5	13	22	9
59	11	1	3	8	4	1
60	11	0	10	13	18	11
61	11	1	10	10	11	15
62	11	1	13	16	20	4
63	11	1	12	20	25	13
64	11	0	11	9	11	9
65	11	1	4	16	15	5
66	11	0	13	16	22	19
67	11	0	8	8	26	7
68	11	1	14	20	25	15

ID	AGE	SEX	SK4-1	SK4-2	SK6-1	SK6-2
69	11	1	15	22	29	27
70	11	0	12	17	23	26
71	11	0	12	12	22	19
72	11	1	14	9	21	11
73	11	1	2	12	23	10
74	11	0	12	13	24	20
75	11	1	13	22	28	27
76	11	1	3	4	6	1
77	11	1	11	9	9	16
78	11	1	11	8	20	12
79	11	1	3	9	12	12
80	11	1	12	14	13	18
81	11	1	13	22	7	9
82	11	1	9	14	12	8
83	11	0	11	11	24	13
84	11	0	7	12	16	8
85	11	0	11	13	10	17
86	11	0	13	20	25	25
87	11	0	10	14	19	26
88	11	0	11	9	17	16
89	11	1	13	21	19	16
90	12	0	5	7	11	0
91	12	0	12	12	18	9
92	12	0	3	5	6	3
93	12	0	2	12	9	3
94	12	1	14	18	10	6
95	12	0	13	19	25	24
96	12	0	3	3	17	5
97	12	1	14	18	25	24
98	12	1	12	9	19	15
99	12	0	14	21	21	23
100	12	0	7	4	22	7
101	12	0	7	6	6	1
102	12	0	9	16	22	22

ID	AGE	SEX	SK4-1	SK4-2	SK6-1	SK6-2
103	12	0	13	13	20	13
104	12	1	14	24	17	22
105	12	0	11	19	19	13
106	12	0	11	23	25	9
107	12	1	14	17	18	22
108	12	0	10	8	24	19
109	12	0	9	15	18	15
110	12	0	8	12	8	21
111	12	0	15	9	26	23
112	12	0	10	19	29	27
113	12	0	14	18	21	9
114	12	0	13	18	22	19
115	12	1	9	17	11	12
116	12	0	8	10	17	13
117	12	1	14	18	17	9
118	12	1	7	18	16	14
119	12	1	11	13	14	11
120	12	1	15	29	28	36
121	12	1	7	8	18	12
122	12	0	12	11	26	24
123	12	1	14	25	23	12
124	12	0	4	23	28	13
125	12	1	13	18	18	7
126	12	1	11	16	16	7
127	12	0	6	14	17	12
128	12	1	1	4	14	3
129	12	1	12	12	11	23
130	12	0	5	3	11	9
131	12	0	13	7	27	12
132	12	0	5	9	7	11
133	12	1	8	18	20	11
134	12	1	10	18	24	17
135	12	0	9	16	21	22
136	12	1	12	21	17	14

ID	AGE	SEX	SK4-1	SK4-2	SK6-1	SK6-2
137	12	0	12	16	29	25
138	12	0	12	14	27	27
139	12	0	15	22	22	22
140	12	0	7	13	18	6
141	12	0	14	20	25	20
142	12	1	14	15	23	21
143	12	0	13	20	27	24
144	13	1	15	5	18	11
145	13	1	8	17	21	17
146	13	0	11	9	18	10
147	13	0	12	19	15	24
148	13	1	12	21	28	16
149	13	0	15	19	23	32
150	13	0	15	15	29	26
151	13	1	12	16	16	26
152	13	0	13	14	22	30
153	13	0	13	20	22	22
154	13	0	12	18	16	8
155	13	0	14	16	22	22
156	13	0	9	12	19	26
157	13	0	10	21	19	13
158	13	1	14	16	22	16
159	13	1	14	26	26	20
160	13	1	12	9	13	9
161	13	0	12	21	22	28
162	13	1	15	16	26	13
163	13	0	11	14	23	4
164	13	0	11	8	20	5
165	13	0	6	15	20	16
166	13	1	10	18	17	16
167	13	0	14	20	28	16
168	13	0	13	19	22	23
169	13	0	15	21	23	25
170	13	1	13	21	29	17

ID	AGE	SEX	SK4-1	SK4-2	SK6-1	SK6-2
171	13	0	13	22	25	26
172	13	0	9	13	14	8
173	13	1	13	16	19	21
174	13	1	12	16	23	31
175	13	1	4	15	7	7
176	13	0	6	8	24	12
177	13	1	12	24	28	29
178	13	1	12	25	18	16
179	13	0	12	21	26	16
180	13	1	12	11	21	11
181	13	1	7	13	22	18
182	13	1	9	19	26	19
183	13	1	11	14	23	18
184	13	0	11	17	26	32
185	13	0	6	11	20	12
186	13	1	4	14	21	14
187	13	0	9	20	11	19
188	13	0	8	15	23	17
189	13	1	14	23	27	25
190	13	1	5	13	19	7
191	13	1	6	16	19	10
192	13	1	15	26	28	31
193	13	1	10	16	19	8
194	14	1	7	8	16	7
195	14	0	6	9	6	10
196	14	0	14	14	18	7
197	14	1	14	27	25	29
198	14	0	3	14	14	6
199	14	0	15	18	24	11
200	14	0	11	11	16	10
201	14	1	3	9	17	11
202	14	0	14	17	20	22
203	14	1	12	21	22	19
204	14	0	2	11	19	23

ID	AGE	SEX	SK4-1	SK4-2	SK6-1	SK6-2
205	14	0	12	8	14	11
206	14	0	14	20	20	6
207	14	1	11	15	12	3
208	14	1	7	12	19	4
209	14	0	4	10	3	1
210	14	0	5	9	16	3
211	14	0	14	25	28	15
212	14	1	10	12	17	10
213	14	0	15	20	20	15
214	14	1	14	20	19	22
215	14	1	13	14	21	14
216	14	1	10	22	23	16
217	14	0	13	8	25	21
218	14	1	10	14	23	11
219	14	1	13	17	24	19
220	14	0	13	10	21	11
221	14	0	9	13	25	17
222	14	0	11	9	19	11
223	14	1	13	17	19	19
224	14	1	13	17	19	14
225	14	1	14	13	19	19
226	14	0	15	18	22	20
227	14	1	4	16	20	18
228	14	1	13	18	28	16
229	14	1	14	16	19	16
230	14	0	5	6	7	20
231	14	1	13	9	22	12
232	14	0	9	10	22	21
233	14	0	15	16	28	26
234	14	1	13	15	22	26
235	14	0	12	18	21	15
236	14	1	8	9	16	18
237	14	1	15	15	27	29
238	14	1	13	12	17	13

ID	AGE	SEX	SK4-1	SK4-2	SK6-1	SK6-2
239	14	1	15	20	27	24
240	15	0	12	13	22	18
241	15	1	11	11	22	14
242	15	0	13	18	27	35
243	15	0	14	22	24	16
244	15	1	9	8	16	5
245	15	1	9	9	25	11
246	15	0	15	19	29	27
247	15	0	5	5	24	13
248	15	0	4	8	25	10
249	15	0	9	16	22	17
250	15	0	12	16	25	6
251	15	0	12	14	19	12
252	15	0	6	15	26	11
253	15	1	15	11	19	17
254	15	1	13	15	23	21
255	15	1	13	15	15	7
256	15	0	11	22	23	11
257	15	0	15	28	29	40
258	15	1	15	22	27	29
259	15	0	15	26	29	25
260	15	1	15	27	29	42
261	15	0	15	17	26	24
262	15	0	14	22	26	30
263	15	0	15	20	28	20
264	15	0	8	11	19	12
265	15	1	15	26	27	41
266	15	1	15	21	28	15
267	15	0	15	20	25	43
268	15	0	14	17	23	21
269	15	1	15	22	21	16
270	15	1	14	23	22	19
271	15	1	15	30	28	34
272	15	1	13	24	22	17

ID	AGE	SEX	SK4-1	SK4-2	SK6-1	SK6-2
273	15	0	14	25	28	28
274	16	0	13	13	16	10
275	16	0	6	11	17	9
276	16	0	10	14	21	8
277	16	0	4	9	23	10
278	16	0	15	26	17	9
279	16	0	15	17	26	26
280	16	0	6	8	4	4
281	16	0	15	24	27	33
282	16	1	13	15	21	20
283	16	1	11	21	21	19
284	16	0	13	20	22	26
285	16	0	15	13	22	16
286	16	0	11	20	19	20
287	16	1	13	30	28	41
288	16	1	13	16	17	14
289	16	1	14	25	23	13
290	16	1	13	20	20	25
291	16	1	13	15	25	41
292	16	1	15	18	29	36
293	16	1	14	23	26	27
294	16	1	14	17	23	13
295	16	1	13	17	24	17
296	16	0	7	6	22	20
297	16	1	13	24	29	18
298	16	1	9	15	25	6
299	16	0	15	22	29	17
300	16	1	13	22	25	28
301	16	0	15	28	28	30
302	16	0	14	18	24	16
303	16	0	4	7	22	27
304	16	0	15	28	28	24
305	16	0	15	22	29	38
306	16	0	15	23	24	16

ID	AGE	SEX	SK4-1	SK4-2	SK6-1	SK6-2
307	16	0	15	24	27	33
308	16	0	13	27	30	26
309	16	1	15	23	28	20
310	16	1	15	26	29	30
311	16	0	13	13	27	33
312	16	0	15	16	29	35
313	16	0	12	15	20	33
314	16	0	5	13	17	17
315	16	0	15	30	23	34
316	16	0	15	25	29	35
317	16	1	13	15	23	14
318	16	0	15	27	28	34
319	16	1	14	13	20	19
320	16	1	15	28	29	33
321	16	1	10	20	25	12
322	16	0	15	30	22	28
323	16	0	15	30	23	26
324	16	0	14	25	25	31
325	16	1	15	21	28	35
326	16	1	12	17	25	25
327	16	1	14	28	25	13
328	16	1	11	17	22	17
329	16	1	14	14	23	18
330	16	1	15	26	28	36
331	16	0	14	20	28	40
332	16	1	14	17	27	21
333	16	1	13	20	28	32
334	16	1	15	20	20	35
335	16	1	8	18	23	15
336	16	1	14	4	26	22
337	16	0	15	14	28	26
338	16	0	14	20	29	35
339	16	0	12	29	26	25
340	16	0	7	22	26	18